Multiplicative Magnitudes*

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Plan

Multiplicative magnitudes: Magnitudes formed by multiplying or dividing magnitudes by others (*length by width*, *miles per hour*).

1. Why multiplicative magnitudes are needed in an ontology for natural language:
   (a) Proportional readings of positive and comparative *much/many*
   (b) Explicitly multiplicative expressions like *miles per hour*
2. Magnitudes in linguistics and metrology
3. Applying ideas from metrology to language

1 Quantity words

Quantity words: *much*, *many*, *few*, *little* (Hoeksema 1983, Partee 1989b, Romero 1998, 2017, Hackl 2000, 2009, Rett 2006, 2008, 2014, 2018, Solt 2009, 2011, 2015, Wellwood 2014, 2015, among many others).

1.1 Desiderata for a theory of quantity words

Wide range of uses.

- quantificational, as in *Many/few students attended the lecture*
- predicative, as in *John's good qualities are many/few*
  (though **I consider John's good qualities few*)
- attributive, as in *The many/few students that we invited enjoyed the lecture*
- differential, as in *Many/few more than 100 students attended the lecture*

*This work has benefitted immensely from discussions with David Alvarez (BA, Boston University, 2019, in linguistics, mathematics and philosophy) who worked with me during the summer of 2019 on this problem. That summer, he established foundations for quantity multiplication based on a homomorphism with the real numbers (Alvarez, 2019). Although I am not relying on the same homomorphism here, discussions with him enabled me to appreciate the more recent work in metrology that my proposal incorporates.

- VP-modificational, as in *Ben does not read books much*
- PP-modificational, as in *The camp is not much beyond the tree line*

Cardinal vs. proportional. Partee (1989a):

(1) There were few faculty children at the 1980 picnic.
Cardinal: Number of children at the picnic may be a high proportion of faculty children, as long as the total number is low.

$$|X \cap Y| < s$$

(2) Few egg-laying mammals suckle their young.
Proportional: Total number of egg-laying mammals that suckle their young is high, but a low proportion of egg-laying mammals.

$$\frac{|X \cap Y|}{X} < p$$

Westerståhl (1985); Herburger (1997):

(3) Few cooks applied.
Reverse proportional reading: The ratio of cook-applicants to applicants was small.

$$\frac{|X \cap Y|}{Y} < p$$

How to account for all these possibilities?

Solt (2009) observes that proportional readings arise with comparatives:

(4) a. More residents of Ithaca than New York City know their neighbors.
b. Fewer Brazilians than Swedes have college degrees.

Bale & Schwarz (2019) add:

(5) a. There are more boats on Lake Ontario than on Lake Superior.
(could be boats per square mile, Lake Superior is 4× bigger)
b. There are more knots in the blue rope than in the red one.
(could be true if both have 20 knots but red is 3× bigger)
c. Your manuscript has more typos than my manuscript.
(could be true if both have 100 types but mine has 10× more words)

Comparison of proportion, not cardinality.

Rarity of proportional readings for quantity superlatives.

- As you may have heard, ordinary gradable adjectives like *tall* are ambiguous in the superlative between ABSOLUTE and RELATIVE readings (Szabolcsi, 1986; Heim, 1999).

(6) Zoe counted to the highest number on *Wednesday*.
'Zoe counted to a higher number on Wednesday than any other day.'
(**Relative reading**)

(7) Zoe beat the hardest boss on Wednesday.
'Zoe beat the boss that was harder than any other boss on Wednesday.'
(**Absolute reading**)

The relative reading seems to involve high scope for *-est* (Heim, 1999) (see Bumford 2017, 2018 for a new and improved scope-based theory).

(8) Wednesday [-est $\lambda d \lambda x$ Zoe counted to a d-high number on x]

(9) Zoe beat the [-est $\lambda d \lambda x$ d-hard boss(x)] on Wednesday

- In the realm of quantity superlatives, the closest thing we find to an absolute reading is a **proportional reading**.

(10) Zoe has visited the most continents.
'Zoe has visited more continents than anyone else has.'
(**Relative reading**)

(11) Zoe has visited most continents.
'Zoe has visited the majority of the continents.'
(**Proportional reading**)

Note: As far as I can see, 'proportional reading' as it applies to quantity superlatives, and stands in contrast to 'relative reading', has nothing to do with the notion of 'proportional reading' that stands in contrast to 'cardinal reading'.[1]

- But quantity superlatives very rarely exhibit proportional readings. E.g. Slovenian:

(12) **Naj-več** ljudi pije pivo.
SPRL-many people drink beer
'More people drink beer than any other beverage.'
(*Unavailable:* 'More than half the people drink beer.')

The probability that a given language will have proportional readings for the superlative form of a quantity word is around 10% (Coppock et al., to appear).

1.2 Some theories of quantity words

1. Quantity words are just like gradable adjectives (e.g. Hoeksema 1983, Teodorescu 2009, Hackl 2009).

(13) *tall* $\rightsquigarrow \lambda d \,.\, \lambda x \,.\, \mathsf{height}(x) \geq d$ $\qquad \langle d, \langle e, t \rangle \rangle$

(14) *much/many* $\rightsquigarrow \lambda d \,.\, \lambda x \,.\, \mu(x) \geq d$ $\qquad \langle d, \langle e, t \rangle \rangle$

[1] Kotek et al. (2012) show that regardless of which *many* you start with, cardinal or proportional, a proportional reading for the superlative *most* can be derived compositionally.

With superlatives, predicts both relative and absolute-like readings.

(15) Zoe has visited (the) most continents.

(16) Zoe [-est $\lambda d \lambda x$ x has visited d-many continents]
'Zoe has visited more continents than anyone else has.' (**relative reading**)

(17) Zoe has visited [-est $\lambda d \lambda x$ d-many continents(x)]
With vanilla *-est*: 'Zoe has visited the continent plurality that is more numerous than any other.' (**totality reading**)
With Hackl's (2009) *-est*: 'Zoe has visited the continent plurality that is more numerous than any non-overlapping one.' (**proportional reading**)

Proportional good for English and German; bad for most languages.

2. Hackl's (2000) semantics for *many*, on which it is a 'parameterized quantificational determiner': a generalized quantifier that also has a degree argument.

(18) $much/many \rightsquigarrow \lambda d \lambda P \lambda Q \,.\, \exists x[P(x) \wedge Q(x) \wedge |x| \geq d]$

Variant (Bale & Schwarz, 2019):

(19) $much/many \rightsquigarrow \lambda d \lambda P \lambda Q \,.\, \mu(P \cap Q) \geq d$
where μ has a contextually set value (e.g. one of μ_{WEIGHT}, μ_{VOLUME}, μ_{LENGTH}, $\mu_{\#}$, $\mu_{\frac{\text{WEIGHT}}{\text{VOL-OF-P}}}$, $\mu_{\frac{\text{WEIGHT}}{\text{VOL-OF-Q}}}$, $\mu_{\frac{\#}{\#\text{-OF-P}}}$, $\mu_{\frac{\#}{\#\text{-OF-Q}}}$, $\mu_{\frac{\#}{\text{LENGTH-OF-ROPE}}}$, $\mu_{\frac{\#}{\text{AREA-OF-LAKE}}}$, etc.

Explains rarity of absolute-like readings (Schwarz, 2004), but not range of uses.

3. Solt (2009)

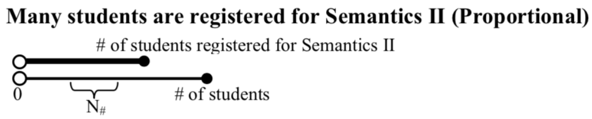
(20) $much/many \rightsquigarrow \lambda d \,.\, \lambda D_{dt} \,.\, D(d)$ $\langle d, \langle dt, t \rangle \rangle$

Explains range of uses (Solt, 2009), as well as rarity of absolute-like readings for quantity superlatives (Coppock et al., to appear): The quantity word has to take sentential scope in *Zoe has visited (the) most continents*.

(21) Zoe [-est $\lambda d \lambda x$ [d-many] $\lambda d'$ x has visited d' MEAS continents]
'Zoe has visited more continents than anyone else has.' (**relative reading**)

Great! How to account for cardinal/proportional ambiguity under this theory?

Solt (2009, 209): "the proportional reading arises when an upper bound to the scale is assumed, whereas the cardinal reading arises when there is no salient upper bound."



2 Rate expressions

Proportional degrees are rates. Expressions that explicitly denote rates:

(22) a. boats per square mile
b. knots per foot
c. cooks per applicant

(23) a. miles per hour
b. situps a day
c. cents on the dollar

Sample challenge problems:

(24) a. Sainetra biked at 15 mph for 2 hours.
b. Therefore, Sainetra biked 30 miles.

(25) a. Sainetra biked 30 miles in 2 hours, at a steady pace.
b. Therefore, Sainetra biked at 15mph for 2 hours.

(26) a. Zahra did 30 situps a day for a week.
b. Therefore, Zahra did 210 situps in one week.

(27) a. Zahra did 210 situps in one week.
b. Zahra did the same number of situps every day that week.
c. Therefore, Zahra did 30 situps a day that week.

(28) a. The professor assigned 1300 pages of reading over the course of the semester.
b. The semester lasted 13 weeks.
c. Therefore, on average, the professor assigned 100 pages of reading per week.

(29) a. Matt drank 1 pint and 9 ounces of beer.
b. A serving of beer is 12 ounces, in the U.S.
c. So, Matt drank about two U.S. servings of beer.

(30) a. The song is 2 minutes long at 180 bpm.
b. Therefore, at 200 bpm the song would be 1 minute and 48 seconds long.

(31) a. I bought this couch for $100 and sold it for 70 cents on the dollar.
b. Therefore I sold the couch for $70.

What do unit expressions like *mile* denote?

Lonning (1987), Champollion (2017):

- measure functions: individuals $\rightarrow$ degrees
(vs. Krifka's (1998) measure functions: individuals $\rightarrow$ numbers)

- unit functions: degrees $\rightarrow$ numbers

Examples from Champollion:

(32) a. $\mathsf{pounds}(\mathsf{weight}(\mathsf{ed})) = 150$
b. $\mathsf{kg}(\mathsf{weight}(\mathsf{ed})) = 68$

(33) a. $\mathsf{feet}(\mathsf{height}(\mathsf{ed})) = 6$
b. $\mathsf{cm}(\mathsf{height}(\mathsf{ed})) = 183$

(34) three inches of oil
a. $\lambda x[\mathsf{oil}(x) \wedge \mathsf{inches}(\mathsf{height}(x)) = 3]$
b. $\lambda x[\mathsf{oil}(x) \wedge \mathsf{inches}(\mathsf{diameter}(x)) = 3$

(35) walk for three hours / three hours of walking
a. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{hours}(\tau(e)) = 3]$

(36) walk for a mile
a. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{miles}(\sigma(e)) = 1]$

3 Multipicative Magnitudes

3.1 Degrees in linguistics and philosophy of language

- Cresswell (1977): Degrees as equivalence classes of (world-bound) objects.
  - Ex. the set of $\langle$world,object$\rangle$ pairs such that the object weighs 6kg in the world

  Addition possible, through concatenation (Klein, 1991).

- Kennedy (1997): degrees are points on a scale ordered along some given dimension.

- Kennedy (2001): Distinction between positive and negative degrees (cf. also Seuren 1978, von Stechow 1984).
  - positive degrees: intervals that range from the lower end of a scale to some point
  - negative degrees: intervals that range from some point to the upper end of a scale.

  Defines a SCALE a "linearly ordered, infinite set of points, associated with a dimension that indicates the type of measurement that the scale represents (e.g. *height*, *length*, *weight*, *brightness* and so forth)."

  Degree: a convex, nonempty subset of a scale.

  (37) ?Alice is shorter than Carmen is tall.

  (38) It is 21 pages long/?short.

- Sassoon (2010) and van Rooij (2011) explicitly discuss multiplication within a particular dimension, building on measurement theory (Krantz et al., 1971).

  (39) John is twice as tall as Mary.

  Calls for RATIO SCALES. (van Rooij points out: "the scale structures invoked in standard degree-based approaches (e.g. Kennedy, 1999, von Stechow, 1984) are generally ratio-scales.")

- Moltmann (2009): Tropes instead of degrees

  (40) John is remarkably tall.

  (41) John's happiness exceeds Mary's.

  Also discusses addition in the style of Klein (1991).

- Anderson & Morzycki (2015), Scontras (2017): Degrees as kinds.

Recent scholarly overview given in Wellwood (2019).

I have not seen any discussion of cross-dimensional multiplication in formal semantics or philosophy of language. (Have I missed something?)

In order to do that, we need an ALGEBRA OF DEGREES.

3.2 International Vocabulary of Metrology (VIM)

Joint Committee for Guides in Metrology (JCGM), formed in 1997, provides a terminological dictionary for METROLOGY, the science of measurement and its application (JCGM, 2012).

quantity: property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference

- r - radius ('GENERIC CONCEPT')
- vs. r_A - radius of circle A ('INDIVIDUAL QUANTITIES')

[r_A: a trope?]

kind of quantity: aspect common to mutually comparable quantities

- The quantities diameter, circumference, and wavelength are generally considered to be quantities of the same kind, namely of the kind of quantity called length.
- The quantities heat, kinetic energy, and potential energy are generally considered to be quantities of the same kind, namely of the kind of quantity called energy.
- Quantities of the same kind within a given system of quantities have the same quantity dimension. However, quantities of the same dimension are not necessarily of the same kind.

> Ex. The quantities moment of force and energy are, by convention, not regarded as being of the same kind, although they have the same dimension. Similarly for heat capacity and entropy, as well as for number of entities, relative permeability, and mass fraction.
>
> Joule per kelvin (J/K) is a measurement unit of both heat capacity and entropy, which are generally not considered to be quantities of the same kind.

[Is this a matter of hyperintensionality?]

system of quantities: set of quantities together with a set of non-contradictory equations relating those quantities

base quantity: quantity in a conventionally chosen subset of a given system of quantities, where no subset quantity can be expressed in terms of the others

derived quantity: quantity, in a system of quantities, defined in terms of the base quantities of that system

> Ex. In a system of quantities having the base quantities length and mass, mass density is a derived quantity defined as the quotient of mass and volume (length to the third power).

International System of Quantities (ISQ): system of quantities based on the seven base quantities:

length	L
mass	M
time	T
electric current	I
thermodynamic temperature	Θ
amount of substance	N
luminous intensity	J

quantity dimension / dimension of a quantity / dimension: expression of the dependence of a quantity on the base quantities of a system of quantities as a product of powers of factors corresponding to the base quantities, omitting any numerical factor

> Ex. In the ISQ, the quantity dimension of force is denoted by $\mathsf{dim}\, F = \mathsf{LMT}^{-2}$.

dimensionless quantity / quantity of dimension one. quantity for which all the exponents of the factors corresponding to the base quantities in its quantity dimension are zero

- Some quantities of dimension one [dimensionless quantities] are defined as the ratios of two quantities of the same kind.

  > Ex. Plane angle, solid angle, refractive index, relative permeability, mass fraction, friction factor, Mach number.

- Numbers of entities are quantities of dimension one.

  Ex. Number of turns in a coil, number of molecules in a given sample, degeneracy of the energy levels of a quantum system.

quantity value: number and reference together expressing magnitude of a quantity.

- E.g. 5.34m or 534cm for length of a given rod [Is this about intensionality?]

numerical value of a quantity: number in the expression of a quantity value, other than any number serving as the reference

- For quantities of dimension one, the reference is a measurement unit which is a number and this is not considered as a part of the numerical quantity value.

  Ex. In an amount-of-substance fraction equal to 3 mmol/mol, the numerical quantity value is 3 and the unit is mmol/mol. The unit mmol/mol is numerically equal to 0.001, but this number 0.001 is not part of the numerical quantity value, which remains 3.

- A quantity can be expressed as the product of a numerical value with a unit:

  $$q = \{q\}[q]$$

  where $\{q\}$ is the numerical value and $[q]$ is the unit.

quantity calculus: set of mathematical rules and operations applied to quantities other than ordinal quantities

4 Quantity Calculus

Raposo (2018, 2019) distinguishes between UNIT-CENTRIC and DIMENSION-CENTRIC approaches to the definition of quantities.

- Unit-centric: A quantity is, by definition, the product of a numerical value (e.g. 6) and a unit (e.g. feet) (Kitano, 2013)
  - starts with a system of units $\{u_1, ..., u_n\}$, including a multiplication operation
  - writes any quantity q in a unique way as:

    $$q = \alpha u_1^{r_1} ... u_n^{r_n}$$

    where α is a real number and $r_1, ..., r_n$ are rational numbers.
- Dimension-centric: A given quantity is inherently associated with a dimension (e.g. length), but not with any particular number or unit.
  - starts with a set of basic dimensions, e.g. $\{\mathsf{L}, \mathsf{T}, \mathsf{M}\}$, which can be multiplied
  - each quantity has a designated dimension

  - quantities can be multiplied with each other in a way that mirrors the multiplicative structure of the dimensions

Problems with Kitano's (2013) approach according to Raposo (2018):

- "[U]nits are the result of an arbitrary agreement and, thus, can be easily changed, while the concept of dimension is more resilient."
- "An algebraic structure for quantity calculus which allows fractional exponents is oversized." Only integer exponents are necessary.

Raposo's (2018) system:

- A group of dimensions $\mathscr{D}$ (a *finitely generated free Abelian group*).
  - $\mathscr{D}$ is a set of dimensions, including basic dimensions and products thereof e.g. $\{\mathsf{L}, \mathsf{T}, \mathsf{M}, \mathsf{L}^2, \mathsf{L}^3, ..., \mathsf{ML}^2, ..., \mathsf{LT}^{-1}, ...\}$
  - Identity element: $1_{\mathscr{D}}$, the dimension of dimensionless quantities.
  - Basis: a finite set of generators, e.g. $\{\mathsf{L}, \mathsf{T}, \mathsf{M}\}$
  - If $\{\mathsf{A}_1, ..., \mathsf{A}_k\}$ is the basis for $\mathscr{D}$ then any element B has a unique expression in terms of the form
$$\mathsf{B} = \mathsf{A}_1^{n_1} \cdots \mathsf{A}_k^{n_k}$$
- A space of quantities $\mathcal{Q}$
  - Quantities in $\mathcal{Q}$ are mapped to their dimensions in $\mathscr{D}$ by a function dim.
    * Ex. If h is Planck's constant than $\mathsf{dim}(h) = \mathsf{L}^2\mathsf{T}^1\mathsf{M}$
  - Quantities of the same dimension can be added and multiplied with each other ($q + r$, $q \cdot r$).
  - The dim mapping is a homomorphism with respect to the product of quantities.
$$\mathsf{dim}(q \cdot r) = \mathsf{dim}(q) \cdot \mathsf{dim}(r)$$
- Units: There is a function σ that chooses for each dimension A a unit $\sigma(\mathsf{A})$, a quantity of dimension A. (Must be a non-zero quantity.)
- Numerical values: The quotent of a quantity and the unit quantity of its dimension, is a dimensionless quantity, which can be regarded as a number, the numerical value of the quantity:
$$v(q) = \frac{q}{\sigma(\mathsf{dim}(q))}$$
  or, more compactly:
$$v(q) = q\sigma(\mathsf{dim}(q))^{-1}$$
  So given a unit system, a quantity q determines a unit $[q]$ and a numerical value $\{q\}$.

5 Application to language (brief and preliminary sketch)

What would *mile* denote?

- A particular quantity (type d), one that serves as $\sigma(\mathsf{L})$ in the given context.
- Or, in Lonning/Champollion style: the value function v that maps a quantity q of dimension L to $\frac{q}{\sigma(\mathsf{L})}$ where $\sigma(\mathsf{L})$ is somehow specified to be 'one mile'

What would *mile per hour* denote?

- $\sigma(\mathsf{L}) \cdot \sigma(\mathsf{T})^{-1}$,
  assuming that $\sigma(\mathsf{L}) = \mathsf{mile}$ and $\sigma(\mathsf{T}) = \mathsf{hour}$. Alternatively:

$$\frac{\sigma(\mathsf{L})}{\sigma(\mathsf{T})}$$

Lexical entry for *per*:[2]

(42) $\lambda u \lambda u' . \frac{u}{u'}$
presupposing that there are dimensions A, B such that $u = \sigma(A)$ and $u' = \sigma(B)$.

(43) walk for three hours
  a. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{hours}(\tau(e)) = 3]$
  b. $\lambda e[{}^*\mathsf{walk}(e) \wedge \tau(e) = q \wedge \mathsf{dim}(q) = \mathsf{T} \wedge \sigma(\mathsf{T}) = \mathsf{hour} \wedge v(q) = 3]$

(44) walk at 5 mph
  a. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{mph}(\mathsf{speed}(e)) = 15]$
  b. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{speed}(e) = q \wedge \mathsf{dim}(q) = \mathsf{LT}^{-1} \wedge \sigma(\mathsf{LT}^{-1}) = \mathsf{mile} \cdot \mathsf{hour}^{-1} \wedge v(q) = 5]$

(45) walk 15 miles
  a. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{miles}(\mathsf{distance}(e)) = 15]$
  b. $\lambda e[{}^*\mathsf{walk}(e) \wedge \mathsf{distance}(e) = q \wedge \mathsf{dim}(q) = \mathsf{L} \wedge \sigma(\mathsf{L}) = \mathsf{mile} \wedge v(q) = 15]$

(46) Axiom: $\forall e : \tau(e) \cdot \mathsf{speed}(e) = \mathsf{distance}(e)$

(47) a. The length by width ratio of this is too big.
  b. $\frac{\mathsf{length}(x)}{\mathsf{width}(x)} > s_{norm}$

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[2] As far as I know, Alvarez (2019) is the only author who has previously offered a lexical entry for *per*. For Alvarez, *per* denotes a function that sends an input to its multiplicative inverse (type $\langle d, d\rangle$, so *per mile* denotes a particular degree). Given an input quantity y, it returns the x such that the product of x and y is the identity element on the 'percent' scale (the scale of dimensionless quantities): $\iota x . x \times y = 1$, or $1/y$. The lexical entry here is slightly different, being of type $\langle d, \langle d, d\rangle\rangle$ (so *per mile* expects a degree argument). This allows it to combine with two arguments via Function Application and result in a degree/quantity.

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