Logic for Busy Professionals

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1 Sets and Relations

Set. A set is an abstract collection of distinct objects which are called the *members* or *elements* of that set. The elements of a set are not ordered, and there may be infinitely many of them or none at all. The elements of set can be specified using *list notation*, e.g.:

$$\{2, 4, 6\}$$

or with *predicate notation* e.g.:

$$\{x \mid x \text{ is a positive even number less than } 7\}.$$

Empty set. The *empty set,* written $\varnothing$ or $\{\}$, is the set containing no elements.

Cardinality. The *cardinality* of a set A, written $|A|$, is the number of elements in A. For example, the cardinality of $\{a, b, c\}$ is 3. The empty set has cardinality 0.

Subset. A is a *subset* of B, written $A \subseteq B$, if and only if every member of A is a member of B.

$$A \subseteq B \text{ iff for all } x\text{: if } x \in A \text{ then } x \in B.$$

Note that the empty set is a subset of every set; since the empty set doesn't have any members, it vacuously satisfies the definition of subset in all cases. (A possibly less confusing way to think of subset in the case of the empty set is as follows: A is a subset of B iff there is no member of A that is not a member of B. There is no member of the empty set at all, so there will never be a member of the empty set that is not a member of B, regardless of what B is.)

Proper subset. A is a *proper subset* of B, written $A \subset B$, if and only if A is a subset of B and A is not equal to B.

$$A \subset B \text{ iff (i) for all } x\text{: if } x \in A \text{ then } x \in B \text{ and (ii) } A \neq B.$$

For example, $\{a, b, c\} \subseteq \{a, b, c\}$ but it is not the case that $\{a, b, c\} \subset \{a, b, c\}$.

Superset. A is a *superset* of B, written $A \supseteq B$, if and only if every member of B is ia member of A.

$$A \supseteq B \text{ iff for all } x\text{: if } x \in B \text{ then } x \in A.$$

Proper superset. A is a *proper superset* of B, written $A \supset B$, if and only if A is a superset of B and A is not equal to B.

$$A \supset B \text{ iff (i) for all } x\text{: if } x \in B \text{ then } x \in A \text{ and (ii) } A \neq B.$$

Set Union. The *union* of A and B, written $A \cup B$, is the set of all entities x such that x is a member of A <u>or</u> x is a member of B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Set Intersection. The *intersection* of A and B, written $A \cap B$, is the set of all entities x such that x is a member of A <u>and</u> x is a member of B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Set Difference. The *difference* of A and B, written $A - B$, is the set of all entities x such that x is an element of A and x is not an element of B.

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

This is also known as the *relative complement* of A and B, or the result of *subtracting* B from A. $A - B$ can also be read, 'A *minus* B'.

Set complement. The *complement* of a set A, written A', is the set of all entities x such that x is not in A.

$$A - B = \{x \mid x \notin A\}$$

Exercise. Given the following sets:

$A = \{a, b, c, 2, 3, 4\}$ $\quad E = \{a, b, \{c\}\}$
$B = \{a, b\}$ $\quad F = \varnothing$
$C = \{c, 2\}$ $\quad G = \{\{a, b\}, \{c, 2\}\}$
$D = \{b, c\}$

classify each of the following statements as true or false.

(a) $c \in A$	(g) $D \subset A$	(m) $B \subseteq G$
(b) $c \in F$	(h) $A \subseteq C$	(n) $\{B\} \subseteq G$
(c) $c \in E$	(i) $D \subseteq E$	(o) $D \subseteq G$
(d) $\{c\} \in E$	(j) $F \subseteq A$	(p) $\{D\} \subseteq G$
(e) $\{c\} \in C$	(k) $E \subseteq F$	(q) $G \subseteq A$
(f) $B \subseteq A$	(l) $B \in G$	(r) $\{\{c\}\} \subseteq E$

Ordered pair. As stated above, sets are not ordered:

$$\{a, b\} = \{b, a\}$$

But the elements of an *ordered pair* written $\langle a, b\rangle$ are ordered. Here, a is the *first member* and b is the *second member*.

$$\langle a, b\rangle \neq \langle b, a\rangle$$

The notion of ordered pair can be defined in terms of the notion of set. We can define $\langle a, b\rangle$ as $\{a, \{a, b\}\}$. Alternatively, we can think of ordered pairs as a separate primitive.

Relation. Transitive verbs like *love*, *admire*, and *respect* are sometimes thought of as *relations* between two individuals. Certain nouns like *mother* can also be thought of as expressing relations.

Relations can be treated as **a set of ordered pairs**. For example, the *love* relation can be treated as the set of ordered pairs of individuals such that the first member loves the second member.

A relation has a domain and a range. The **domain** is the set of objects from which the first members are drawn, and the **range** is the set of objects from which the second members are drawn.

Function. A function is a special kind of relation. A relation R from A to B is a function if and only if it meets both of the following conditions:

- Each element in the domain is paired with just one element in the range.
- The domain of R is equal to A

A function can be thought of as something that gives a single **output** for a given **input**.

For example, $\{\langle 0, 0\rangle, \langle 0, 1\rangle\}$ is not a function because given the input 0, there are two outputs: 0 and 1.

But $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ is a function. Another notation for the same function:

$$\begin{bmatrix} 0 & \rightarrow & 1 \\ 1 & \rightarrow & 0 \end{bmatrix}$$

We write $f(a)$ to denote 'the result of applying function f to argument a' or f of a' or 'f applied to a'. If f is a function that contains the ordered pair $\langle a, b\rangle$, then:

$$f(a) = b$$

This means that given a as input, f gives b as output. E.g. $f(0) = 1$ above.

Characteristic function of a set. A function that yields 1 ("true") for every input that is in set S and 0 ("false") for every input that is not in S is called the **characteristic function** of S.

Exercise. Consider the following functions from the Simpsons to truth values:

$f_1 = \{\langle \text{Homer}, 0\rangle, \langle \text{Marge}, 0\rangle, \langle \text{Bart}, 1\rangle, \langle \text{Lisa}, 1\rangle, \langle \text{Maggie}, 1\rangle\}$
$f_2 = \{\langle \text{Homer}, 1\rangle, \langle \text{Marge}, 1\rangle, \langle \text{Bart}, 0\rangle, \langle \text{Lisa}, 0\rangle, \langle \text{Maggie}, 0\rangle\}$

1. What value does f_1 take, given Marge as an argument? In other words, what is $f_1(\text{Marge})$?
2. What is $f_2(\text{Marge})$?
3. Which is the characteristic function of the set $\{\text{Bart}, \text{Lisa}, \text{Maggie}\}$?

2 Logic

2.1 Propositional Logic (PL)

2.1.1 Syntax

Vocabulary: Set of proposition letters: P, Q, R, S.

Syntax of PL (Definition)

1. Any proposition letter is a formula (an atomic formula)
2. If ϕ is a formula, then $\neg\phi$ is a formula
3. If ϕ and ψ are formulas, then $[\phi \wedge \psi]$ is a formula
4. If ϕ and ψ are formulas, then $[\phi \vee \psi]$ is a formula
5. If ϕ and ψ are formulas, then $[\phi \rightarrow \psi]$ is a formula
6. If ϕ and ψ are formulas, then $[\phi \leftrightarrow \psi]$ is a formula

Which of the following are formulas of PL?

1. $[\neg][\wedge PQ \rightarrow$
2. ϕ
3. $\neg P$
4. $\neg\neg\neg\neg\neg P$
5. $P \wedge Q$
6. $[P \wedge Q]$
7. $[\neg P \wedge [Q]]$
8. $[\neg P \wedge [Q \vee [R \rightarrow S]]]$

2.1.2 Semantics

There are two possible semantic values for a formula in PL:

- 1 (True)
- 0 (False)

(Sometimes people use three truth values, including an *undefined* value, or four, adding an *over-defined* value.)

A **model** (or **structure**) for PL assigns truth values to all of the proposition letters. In PL, a model is just a function that takes a proposition letter and returns 1 or 0. For example, it might be the case in model M_1 that P is true, Q is false, R is true and S is false.

$$M_1 = \begin{bmatrix} P & \rightarrow & 1 \\ Q & \rightarrow & 0 \\ R & \rightarrow & 1 \\ S & \rightarrow & 0 \end{bmatrix}$$

Thus $M_1(P) = 1$, $M_1(Q) = 0$, $M_1(R) = 1$ and $M_1(S) = 0$.

Semantics of PL (Definition)

1. If ϕ is an atomic formula, then $[\![\phi]\!]^M = M(\phi)$
2. $[\![\neg\phi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 0$
3. $[\![\phi \wedge \psi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$
4. $[\![\phi \vee \psi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 1$ or $[\![\psi]\!]^M = 1$
5. $[\![\phi \rightarrow \psi]\!]^M = 1$ unless $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 0$
6. $[\![\phi \leftrightarrow \psi]\!]^M$ if and only if $[\![\phi]\!]^M = [\![\psi]\!]^M$

Truth in a model. A formula ϕ is true in a model M, written:

$$M \vDash \phi$$

if and only if $[\![\phi]\!]^M = 1$.

Exercise. Specify a model M_2 for PL in which $[P \rightarrow Q]$ is false, and another model M_3 in which it is true, and explain why the formula is false in M_2 but true in M_3.

2.2 First-order logic without variables (L_1)

Now we will define a language that has things like verbs and proper names, called L_1.

2.2.1 Syntax

Vocabulary of L_1 (Definition)

1. individual constants/terms: D, N, J, M
2. unary predicates: HAPPY, BORED
3. binary predicates: KISSED, LOVES

Syntax of L_1 (Definition)

1. If π is a unary predicate and α is a term, then $\pi(\alpha)$ is a formula.
2. If π is a binary predicate and α and β are terms, then $\pi(\alpha, \beta)$ is a formula.
3. If ϕ is a formula, then $\neg\phi$ is a formula.
4. If ϕ is a formula and ψ is a formula, then $[\phi \wedge \psi]$ is a formula.
5. If ϕ is a formula and ψ is a formula, then $[\phi \vee \psi]$ is a formula.
6. If ϕ is a formula and ψ is a formula, then $[\phi \rightarrow \psi]$ is a formula.
7. If ϕ is a formula and ψ is a formula, then $[\phi \leftrightarrow \psi]$ is a formula.

Exercise. Which of the following are formulas of L_1?

1. ¬¬HAPPY(M)
2. HAPPY(A)
3. HAPPY(M,J)
4. LOVES(J,M) ∨ LOVES(M,M)
5. [LOVES(J,M) ↔ LOVES(M,J)]
6. [KISSED(M) ↔ LOVES(M,J)]
7. ¬[LOVES(J,M) ← ¬LOVES(M,J)]

2.2.2 Semantics

First-order models. The semantics of L_1 will be given in terms of a first-order model with two components:

$$M = \langle D, I \rangle$$

Such a model consists of a **domain** of individuals D and an **interpretation function** I which specifies the values of all of the expressions in the vocabulary.

The interpretation function I will assign values to all of the expressions in the vocabulary, including terms, unary predicates, and binary predicates.

- The interpretation of an individual constant will be an individual in D.
- The interpretation of a unary predicate will be a set of individuals.
- The interpretation of a binary predicate will be a binary relation between individuals: a set of ordered pairs.

Semantics of $\mathbf{L}_1$ (Definition)

1. If α is a predicate or a term, then $[\![\alpha]\!]^M = I(\alpha)$.
2. If π is a unary predicate and α is a term, then:
   - $[\![\pi(\alpha)]\!]^M = 1$ if $[\![\alpha]\!]^M \in [\![\pi]\!]^M$
   - $[\![\pi(\alpha)]\!]^M = 0$ otherwise.
3. If π is a binary predicate and α and β are terms, then:
   - $[\![\pi(\alpha,\beta)]\!]^M = 1$ if $\langle [\![\alpha]\!]^M, [\![\beta]\!]^M \rangle \in [\![\pi]\!]^M$
   - $[\![\pi(\alpha,\beta)]\!]^M = 0$ otherwise.
4. $[\![\neg\phi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 0$
5. $[\![\phi \wedge \psi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.
6. $[\![\phi \vee \psi]\!]^M = 1$ if and only if $[\![\phi]\!]^M = 1$ or $[\![\psi]\!]^M = 1$.
7. $[\![\phi \rightarrow \psi]\!]^M = 1$ unless $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 0$.
8. $[\![\phi \leftrightarrow \psi]\!]^M$ if and only if $[\![\phi]\!]^M = [\![\psi]\!]^M$.

Exercise. Choose a model M and compute $[\![\textsc{bored}(\textsc{d})]\!]^M$.

2.3 Full first-order logic (FOL)

Predicate calculus (a.k.a. first-order logic) famously has sentences with quantifiers and variables like these:

$\forall x$ HAPPY(x) 'for all x, x is happy'
$\exists x$ ¬HAPPY(x) 'there exists an x such that it is not the case that x is happy'

We ignored them in the previous section, and now we will add them to our language. We will call our new language FOL 'first-order logic'.

2.3.1 Syntax

We will allow an infinite number of variables $v_0, v_1, v_2, ...$ ranging over individuals, but use the following shorthands:

- x is v_0
- y is v_1
- z is v_2

Now we will have two kinds of **terms** (i.e., individual-denoting expressions):

- **(individual) variables** like x, y and z
- **individual constants** like D and N

Not all constants denote individuals; unary and binary predicates also count as constants, as do symbols like $\vee$ and $\neg$. But there are two different kinds of constants:

- **logical constants**: like $\vee$ and $\neg$, whose interpretation does not depend on the model
- **non-logical constants**: like D and HAPPY, whose interpretation depends on the model

Along with the variables, we will add two new logical constants: the universal quantifier $\forall$, and the existential quantifier $\exists$.

Syntax of FOL (Definition)

1. If π is a unary predicate and α is a term, then $\pi(\alpha)$ is a formula.
2. If π is a binary predicate and α and β are is a terms, then $\pi(\alpha,\beta)$ is a formula.
3. If ϕ is a formula, then $\neg\phi$ is a formula.
4. If ϕ is a formula and ψ is a formula, then $[\phi \wedge \psi]$ is a formula.
5. If ϕ is a formula and ψ is a formula, then $[\phi \vee \psi]$ is a formula.
6. If ϕ is a formula and ψ is a formula, then $[\phi \rightarrow \psi]$ is a formula.
7. If ϕ is a formula and ψ is a formula, then $[\phi \leftrightarrow \psi]$ is a formula.
8. If u is a variable and ϕ is a formula, then $\forall u\phi$ is a formula.
9. If u is a variable and ϕ is a formula, then $\exists u\phi$ is a formula.

Free variables. A **free variable** is a variable that is not bound by any quantifier. A formula is called **open** if it contains free variables. Otherwise, it is **closed**.

	Formula	variables	free variables	open or closed?
a.	HAPPY(M)	-	-	closed
b.	HAPPY(x)	x	x	open
c.	$\exists x$ HAPPY(x)	x	-	closed
d.	$\exists y$ HAPPY(x)	x, y	x	open
e.	$\exists y$ LOVES(x, y)	x, y	x	open
f.	$\forall x \forall y$ LOVES(x, y)	x, y	-	closed
g.	[$\forall y$ LOVES(y, x) $\wedge$ HAPPY(z)]	x, y, z	x, z	open

Here is how to calculate the free variables in any formula:

1. If ϕ is an atomic formula, then the set of free variables in ϕ is the set of variables in ϕ.
2. The set of free variables in $\neg\phi$ is the set of free variables in ϕ.
3. The set of free variables in $\phi \wedge \psi$, $\phi \vee \psi$, or $\phi \rightarrow \psi$ is the set of variables that are free in ϕ or ψ.
4. The set of free variables in $\forall v \phi$ or $\exists v \phi$ is the set of free variables in ϕ, excluding v (because v is now bound).

2.3.2 Semantics

Informally, $\forall x$ HAPPY(x) is true in M iff no matter which individual we assign to x, HAPPY(x) is true in M. In other words, for all elements in the domain d, it holds that d satisfies the description HAPPY(x) in M.

Likewise, informally, $\exists x$ HAPPY(x) is true iff we can find some individual to assign to x such that HAPPY(x) is true. In other words, there is some element in the domain d that satisfies the description HAPPY(x) in M.

And $\exists x \exists y$ LOVES(x, y) is true iff we can find a pair of individuals d and d' that satisfy the description LOVES(x, y).

This leads to the idea of an *assignment* of variables to objects. Technically, an **assignment** is a function that assigns individuals to variables. Here are some examples of assignment functions:

$$g_1 = \begin{bmatrix} x & \rightarrow & \text{Maggie} \\ y & \rightarrow & \text{Bart} \\ z & \rightarrow & \text{Bart} \\ \dots \end{bmatrix} \qquad g_2 = \begin{bmatrix} x & \rightarrow & \text{Bart} \\ y & \rightarrow & \text{Bart} \\ z & \rightarrow & \text{Bart} \\ \dots \end{bmatrix}$$

So now, in general, we will make interpretation relative to a model *and an assignment*. Instead of

$$[\![\phi]\!]^M$$

we will now write:

$$[\![\phi]\!]^{M,g}$$

where g stands for an assignment.

The denotation of the variable x with respect to model M and assignment function g is simply whatever g maps x to. (The denotation of the variable depends only on the assignment function, and not on the model.) We can express this more formally as follows:

$$[\![x]\!]^{M,g} = g(x)$$

For example, $[\![x]\!]^{M,g_1} = g_1(x)$ =Maggie, and $[\![x]\!]^{M,g_2} = g_2(x)$ =Bart.

The other semantic rules that we had for L_1 will be the same in FOL, except that the denotation is always relative not only to a model M but also to an assignment g. Furthermore, we will add rules for universal and existential quantification.

Semantics of FOL (Definition)

1. If α is a non-logical constant, then $[\![\alpha]\!]^{M,g} = \text{I}(\alpha)$.
2. If α is a variable, then $[\![\alpha]\!]^{M,g} = g(\alpha)$.
3. If π is a unary predicate and α is a term, then $[\![\pi(\alpha)]\!]^{M,g} = 1$ if $[\![\pi]\!]^{M,g} \in [\![\alpha]\!]^{M,g}$ and 0 otherwise.
4. If π is a binary predicate and α and β are terms, then $[\![\pi(\alpha,\beta)]\!]^{M,g} = 1$ if $\langle [\![\alpha]\!]^{M,g}, [\![\beta]\!]^{M,g} \rangle \in [\![\pi]\!]^{M,g}$ and 0 otherwise.
5. $[\![\neg\phi]\!]^{M,g} = 1$ if $[\![\phi]\!]^{M,g} = 0$; otherwise $[\![\neg\phi]\!]^{M,g} = 0$.
6. $[\![\phi \wedge \psi]\!]^{M,g} = 1$ if $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$; 0 otherwise. Similarly for $[\![\phi \vee \psi]\!]^{M,g}$, $[\![\phi \rightarrow \psi]\!]^{M,g}$, and $[\![\phi \leftrightarrow \psi]\!]^{M,g}$.
7. $[\![\forall v \phi]\!]^{M,g} = 1$ iff for all $d \in D$, $[\![\phi]\!]^{M,g'} = 1$, where g' is an assignment function exactly like g except that $g'(v) = d$.
8. $[\![\exists v \phi]\!]^{M,g} = 1$ iff there is a $d \in D$ such that $[\![\phi]\!]^{M,g'} = 1$, where g' is an assignment function exactly like g except that $g'(v) = d$.

Fact: The truth of a formula with a free variable will depend on the assignment function. The denotation of a formula with no free variables doesn't depend on the assignment function.