Triangle equivalences

Liz Coppock*

Boston University

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*Special thanks to Anastasiia Tatlubaeva, a great undergraduate research assistant who observed the pattern last fall and inspired me to analyze it.

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Triangle Equivalences

(1) a. The cost of wheat is [$100 per ton].
b. The [cost of wheat per ton] is $100.

(2) a. a shortfall of [100 billion euros per annum]
b. a [per annum shortfall] of 100 billion euros

Triangle Equivalence: Definition

A triangle equivalence is a pair of equivalent sentences where:

- In one, a specific quantity (or 'degree', type d), usually expressed by a cardinal number and a unit-denoting noun, merges with the denominator phrase. The combination is linked via a copular relation (e.g. *be*, *of*) with a noun denoting a measure function ($\langle e, d\rangle$; a μ-noun).

$$\mu \simeq (\text{quantity} \div \text{denominator})$$

- In the other, the denominator phrase merges with a μ-noun. This combination is linked via a copular relation to a specific quantity.

$$(\mu \ominus \text{denominator}) \simeq \text{quantity}$$

English-Polish (EuroParl)

(3) a. The price is still going to be USD 80-100 per barrel.

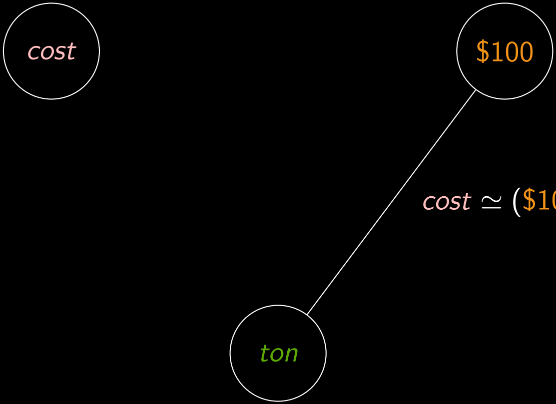
$$\mu \simeq (\text{quantity} \div \text{denominator})$$

b. Cen-a za baryłk-ę będzie nadal
price-nom for barrel-acc be.fut.3.sg still
wynosi-ł-a 80-100 dolar-ów.
amount.to-prt-3.sg.f 80-100 dollars-gen

$$(\mu \ominus \text{denominator}) \simeq \text{quantity}$$

Thanks to Anastasiia Tatlubaeva for the example and gloss.

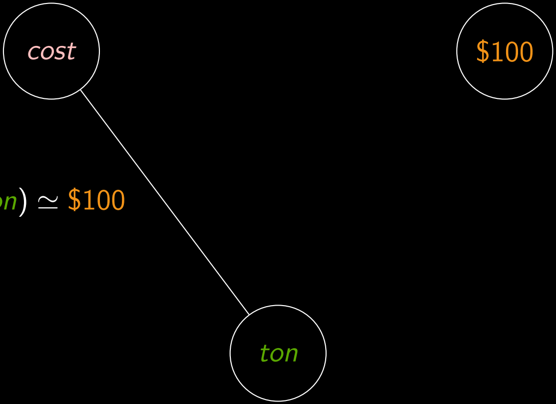
A Quotient Triangle



$\textit{cost} \simeq (\$100 \div \textit{ton})$

The cost is $100 per ton

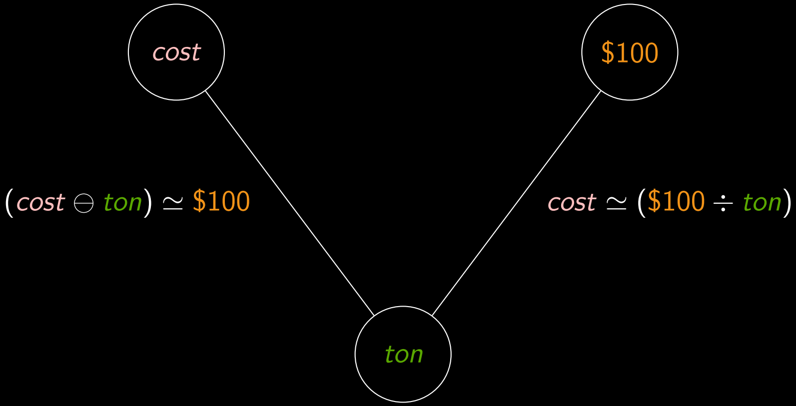
A Quotient Triangle



$(cost \ominus ton) \simeq \$100$

The cost per ton is $100

A Quotient Triangle



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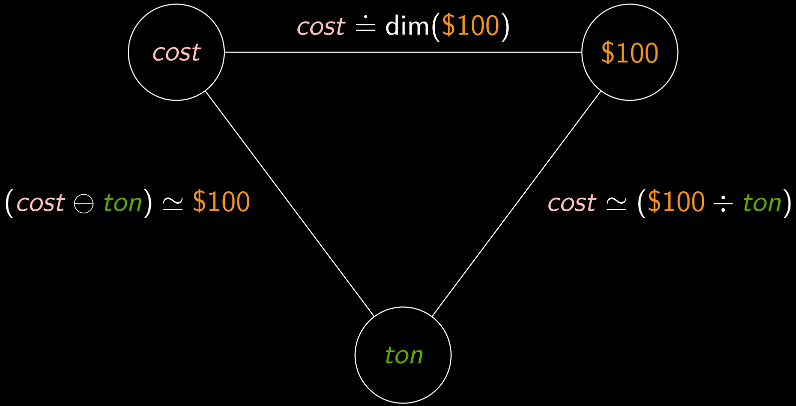
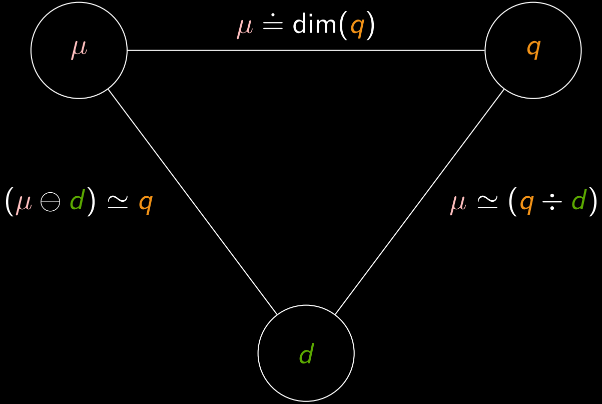


The Quotient Triangle



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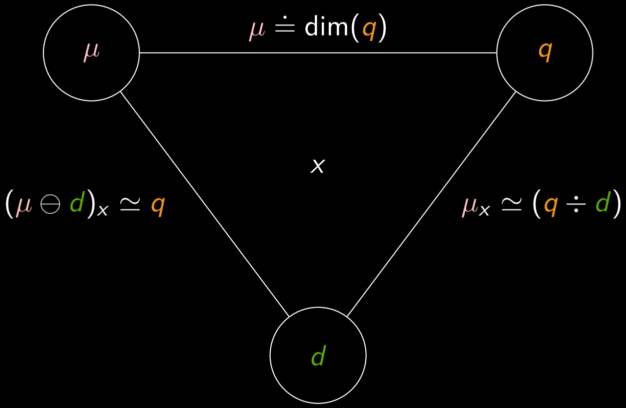

What lies behind this Quotient Triangle?

What coherent interpretations could we give to $\ominus$ and $\div$?

For $(\mathit{cost} \ominus \mathit{ton})_x \simeq \$100$:

What is a term involving *cost* and *ton* that is actually equal to \$100?

$\dim(\$100) = \text{money}$

For $\mathit{cost}_x \simeq (\$100 \div \mathit{ton})$:

What quantity involving *cost* is actually equal to $\frac{\$100}{\mathit{ton}}$?

$\mathit{dim}(\frac{\$100}{\mathit{ton}}) = \frac{\text{money}}{\text{weight}}$

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Quantity calculus: the study of quantities

quantity: property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference, e.g.:

- radius (of circle A), wavelength (of the sodium D radiation)
- kinetic energy, heat
- electric charge, electric resistance

Joint Committee for Guides in Metrology (JCGM, 2012)

Quantity calculus

Three operations:

- product of quantities
- product of a number times a quantity
- addition of quantities of the same kind

(often presented as important starting point)

Quantity calculus

- History goes back to Fourier 1822 (de Boer, 1994)

Quantity calculus

- History goes back to Fourier 1822 (de Boer, 1994)
- Two approaches to the algebraic foundations:
  - **Unit-centric**: e.g. Carlson 1979, Kitano 2013
  - **Dimension-centric**: e.g. Krystek 2015, Raposo 2018, 2019

  "Under this viewpoint, the dimension is an intrinsic property of a quantity, in contrast to its numerical value, which depends on the unit chosen, or the unit itself, which can be changed arbitrarily."

Basic dimensions ($\mathscr{B}$)

Dimension	**Base unit**
L – length	meter (m)
M – mass	kilogram (kg)
T – time	second (s)
I – electric current	ampere (A)
Θ – thermodynamic temperature	kelvin (K)
N – amount of substance	mole (mol)
J – luminous intensity	candela (cd)

(JCGM, 2012)

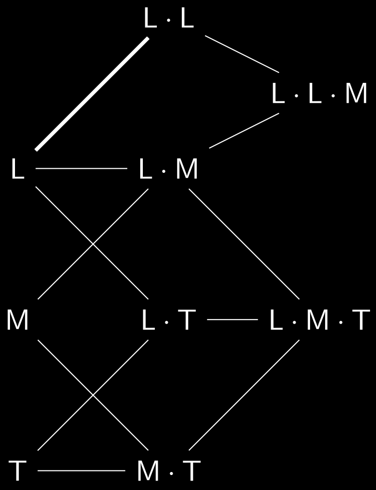
Derived dimensions

Example: If h is Planck's constant then

$$\mathsf{dim}(h) = \mathsf{M} \cdot \mathsf{L}^2 \cdot \mathsf{T}^{-1}$$

Fact:
The Planck constant multiplied by a photon's frequency is equal to a photon's energy.

Derived dimensions



The dimensions form a group

$\mathscr{D}$ is a group, so:

- if $\mathsf{A}, \mathsf{B} \in \mathscr{D}$, then $\mathsf{A} \cdot \mathsf{B} \in \mathscr{D}$
- $\mathscr{D}$ has an identity element $\mathbf{1}_{\mathscr{D}}$, such that for every $\mathsf{D} \in \mathscr{D}$:

$$\mathsf{D} \cdot \mathbf{1}_{\mathscr{D}} = \mathbf{1}_{\mathscr{D}} \cdot \mathsf{D} = \mathsf{D}$$

- There is an inverse D^{-1} for every $\mathsf{D} \in \mathscr{D}$:
  an element such that

$$\mathsf{D} \cdot \mathsf{D}^{-1} = \mathbf{1}_{\mathscr{D}}$$

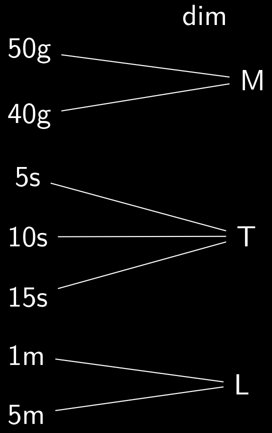
Dimension mapping

$$\mathscr{Q} \xrightarrow{\mathsf{dim}} \mathscr{D}$$

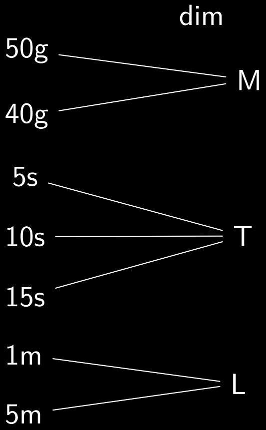
for any quantity Q $\in \mathscr{Q}$:

dim(Q) $\in \mathscr{D}$

Dividing up the quantities by dimension

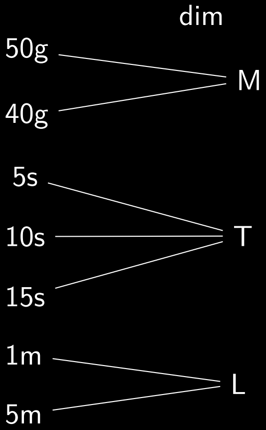


Dividing up the quantities by dimension



$\text{dim}^{-1}(\text{M})$
the set of quantities of mass

Dividing up the quantities by dimension



$\mathrm{dim}^{-1}(\mathrm{M})$
the set of quantities of mass

Notation:
$\mathrm{dim}^{-1}(\mathrm{D}) = \mathscr{Q}_{\mathrm{D}}$

Dimensionless quantities

So-called "dimensionless quantities" have dimension $\mathbf{1}_{\mathscr{D}}$:

- ratios of two quantities of the same kind
  - Ex. relative permeability, dollars earned per dollars saved
- numbers of entities (cardinalities)
  - Ex. Number of molecules in a given sample

(JCGM, 2012)

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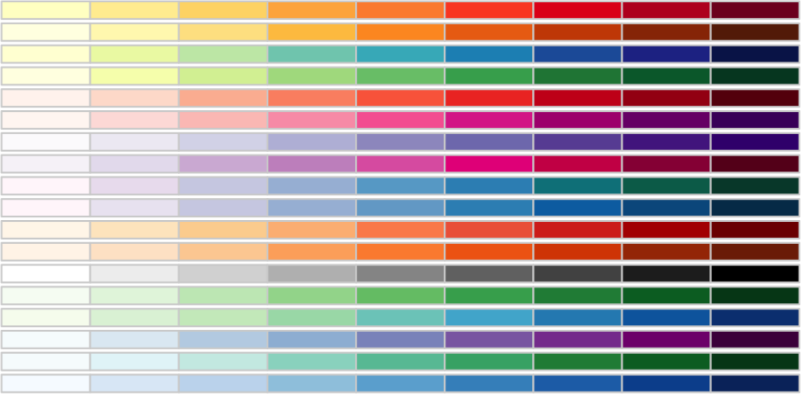
Fiber bundle



Each fiber is a vector space over $\mathbb{R}$

For all dimensions D, $\mathscr{Q}_{\mathsf{D}}$ is a vector space over $\mathbb{R}$, so:

- There exists a **zero element** $\mathbf{0}_{\mathsf{D}} \in \mathscr{Q}_{\mathsf{D}}$ such that for any $q \in \mathscr{Q}_{\mathsf{D}}$:

$$q + \mathbf{0}_{\mathsf{D}} = q$$

- For any $q \in \mathscr{Q}_{\mathsf{D}}$, there exists an **additive inverse element** $-q \in \mathscr{Q}_{\mathsf{D}}$ such that:

$$q + (-q) = \mathbf{0}_{\mathsf{D}}$$

- There exists a **multiplicative identity element 1** from $\mathbb{R}$ such that for any $q \in \mathscr{Q}_{\mathsf{D}}$:

$$q * \mathbf{1} = \mathbf{1} * q = q$$

Cross-dimensional multiplication

$\langle \mathscr{Q}, * \rangle$ is an **abelian monoid**, so:

- If $q_1, q_2 \in \mathscr{Q}$, then $q_1 * q_2 \in \mathscr{Q}$
- There is a **multiplicative identity** element $\mathbf{1}$ such that for all $q \in \mathscr{Q}$:

$$q * \mathbf{1} = \mathbf{1} * q = q$$

- If $q_1, q_2, q_3 \in \mathscr{Q}$ then $q_1 * (q_2 * q_3) = (q_1 * q_2) * q_3$ (**associativity**)
- $q_1 * q_2 = q_2 * q_1$ (**commutativity**)

Existence of inverses

Not every quantity has an inverse; you can't divide by any $\mathbf{0}_{\mathsf{D}}$ ($\mathsf{D} \in \mathscr{D}$).

But for every *non-zero* quantity $q \in \mathscr{Q}$
there is an inverse q^{-1}:

$$q * q^{-1} = \mathbf{1}$$

Or: The set of non-zero quantities forms a group under multiplication.

Unit mapping

$$\mathscr{Q} \xleftarrow{\text{unit}} \mathscr{D}$$

where unit(D) picks out a $q \in \mathscr{Q}_{\text{D}}$
(a q such that $\dim(q) = \text{D}$)

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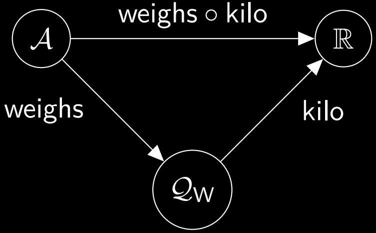
Restrictions:

- You can't pick the zero element.
- unit must be a group homomorphism:

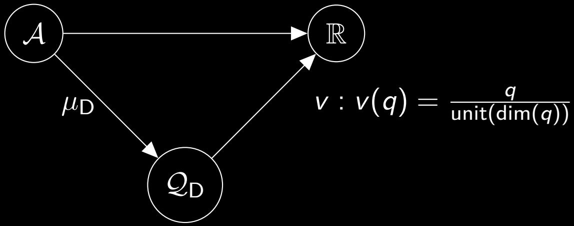
$$\text{unit}(\mathrm{A} \cdot \mathrm{B}) = \text{unit}(\mathrm{A}) * \text{unit}(\mathrm{B})$$

The Lønning Triangle



(Lonning, 1987; Champollion, 2017)

The Lønning Triangle (new variant)



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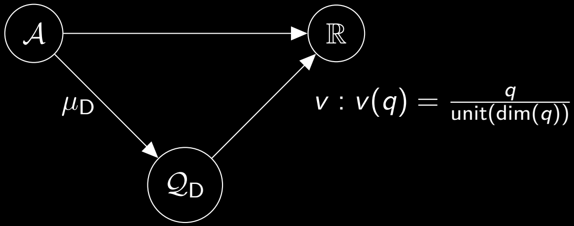

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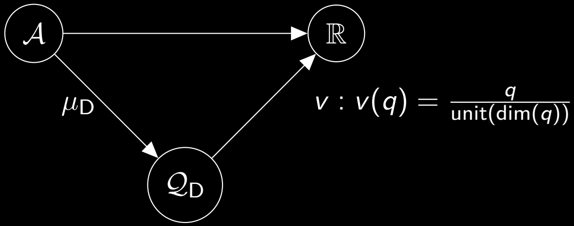

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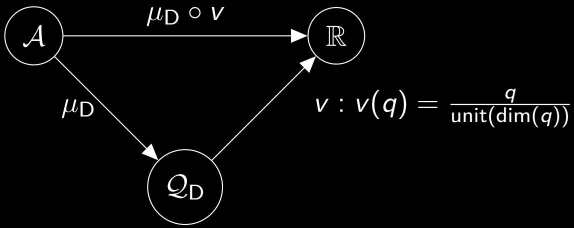

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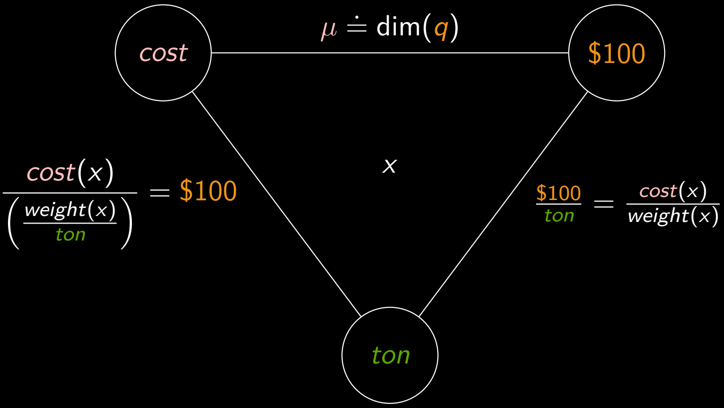

The dimensions match

$$\frac{\textit{cost}(x)}{\left(\frac{\textit{weight}(x)}{\textit{ton}}\right)} = \$100$$

$$\dim(\$100) = \textit{money}$$

$$\dim\left(\frac{\textit{cost}(x)}{\left(\frac{\text{weight}(x)}{\textit{ton}}\right)}\right) = \frac{\textit{money}}{\left(\frac{\textit{weight}}{\textit{weight}}\right)} = \frac{\textit{money}}{1} = \textit{money}$$

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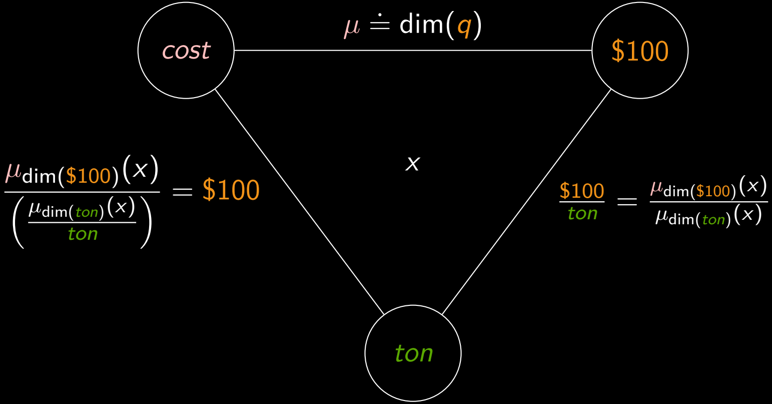

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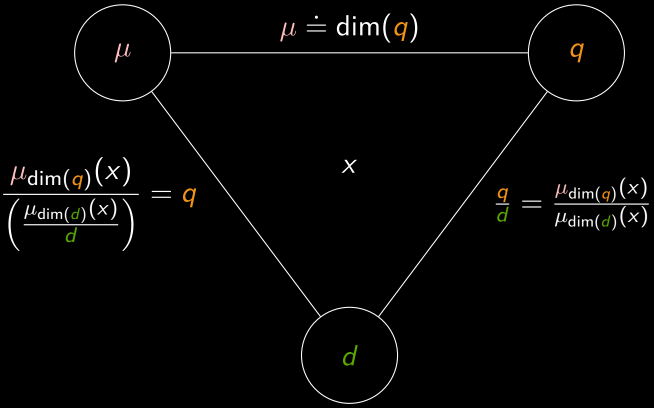

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Quotient function analysis of *per*

Lexical entries for English words:

(4) *kilometer(s)* $\rightsquigarrow$ km

(5) *hour(s)* $\rightsquigarrow$ hour

km, hour: type *d*

Quotient function analysis of *per*

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(6) $per_0 \rightsquigarrow \lambda d \lambda q \, . \, \frac{q}{d}$

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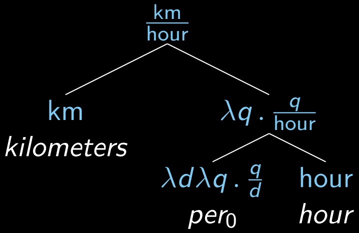

(6) $per_0 \rightsquigarrow \lambda d \lambda q \,.\, \frac{q}{d}$

My 2021 SALT paper. Other mentions of cross-dimensional degree division: Solt (2009), Rawlins (2013), Bale & Schwarz (2019, 2020). Cf. also Tovena (2016) and the literature on *percent* (Ahn & Sauerland, 2015; Li, 2018; Solt, 2018; Spathas, 2019; Pasternak, 2019; Coppock, to appear, i.a.).

Capturing inferences

(7) Sainetra walked at 5 km per hour for 2.5 hours.
Therefore, Sainetra walked 12.5 km.

Distributivity-marker analysis?

Panaitescu & Tovena (2019) treat Romanian *de* and Italian *per*—possible translations of English *per*—as distributivity markers, meaning something like *for every*.

Licensing of *per* by gradable predicates speaks against this:

(8) a. far cheaper to operate per citizen
  b. greater environmental impact per euro
  c. largest net contributor per citizen

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Challenge for the quotient function analysis

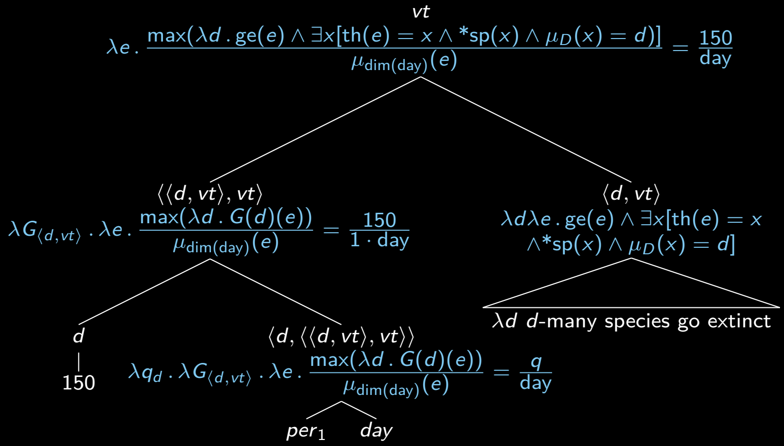
(9) It's estimated that 150 species per day go extinct.
150 species per day is a high rate.
#Therefore, a high rate is among those going extinct.

Quotient operator analysis of *per*

$$per_1 \rightsquigarrow \lambda d_d \,.\, \lambda q_d \,.\, \lambda G_{\langle d,\tau t\rangle} \,.\, \lambda \alpha_\tau \,.\, \frac{\max(\lambda d' \,.\, G(d')(\alpha))}{\mu_{\text{dim}(d)}(\alpha)} = \frac{q}{d}$$

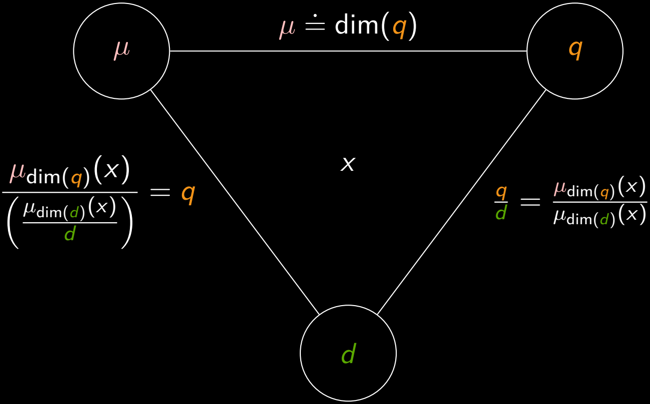
where $\mu_{\text{dim}(d)}(\alpha)$ is the measure of α along the dimension of quantity d.

Cf. Kennedy & Stanley (2009) on *average*.



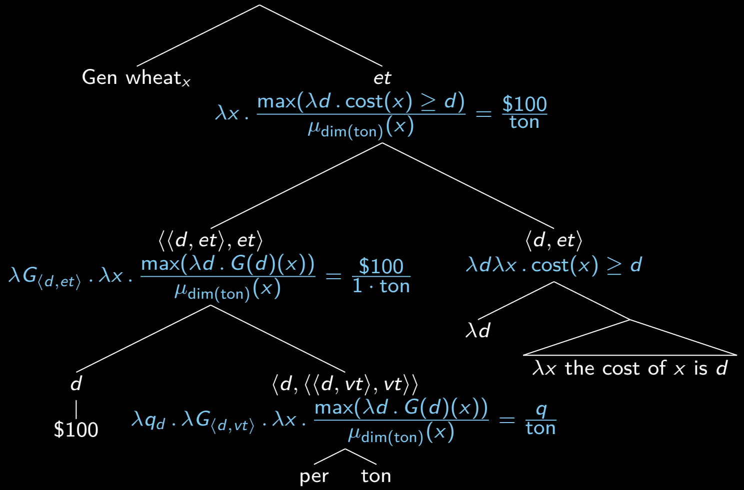
(to be presented at SALT 2022)

What lies behind the Quotient Triangle?



$$per_1 \rightsquigarrow \lambda d_d \,.\, \lambda q_d \,.\, \lambda G_{\langle d, \tau t \rangle} \,.\, \lambda \alpha_\tau \,.\, \frac{\max(\lambda d' \,.\, G(d')(\alpha))}{\mu_{\dim(d)}(\alpha)} = \frac{q}{d}$$

The cost of wheat is \$100 per ton



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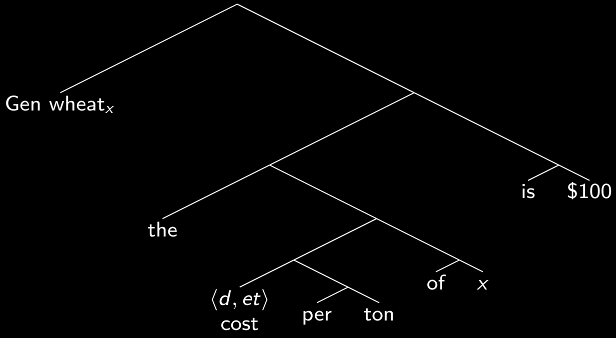

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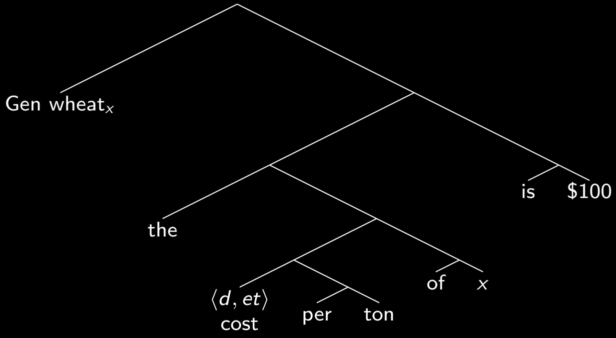

$$per_2 \rightsquigarrow \lambda d_d \,.\, \lambda G_{\langle d, \tau t \rangle} \,.\, \lambda \alpha_\tau \,.\, \lambda q_d \,.\, \frac{\max(\lambda d' \,.\, G(d')(\alpha))}{\mu_{\mathrm{dim}(d)}(\alpha)} = \frac{q}{d}$$

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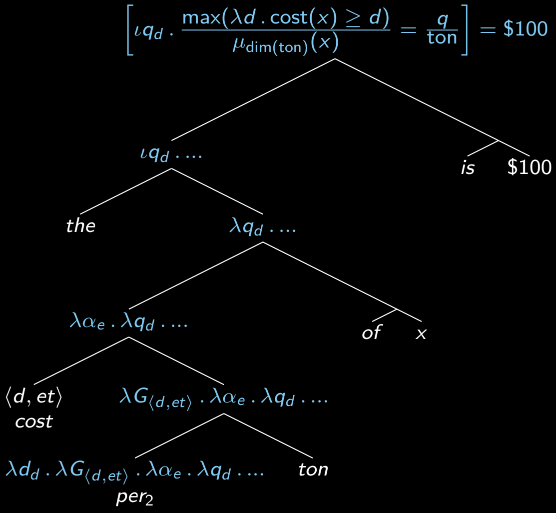

This does give the right result

$$\left[\iota q_d \,.\, \frac{\max(\lambda d\,.\,\text{cost}(x) \geq d)}{\mu_{\text{dim(ton)}}(x)} = \frac{q}{\text{ton}}\right] = \$100$$

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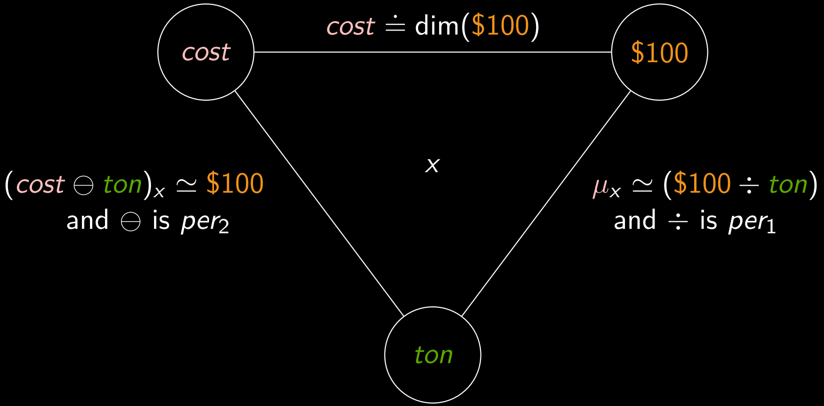
which is another way of saying

$$\frac{\mathrm{cost}(x)}{\left(\frac{\mu_{\mathrm{dim(ton)}}(x)}{\mathrm{ton}}\right)} = \$100$$

Are we done?


$cost \doteq \dim(\$100)$
$cost$
$\$100$
x
$(cost \ominus ton)_x \simeq \$100$
and $\ominus$ is per_2
$\mu_x \simeq (\$100 \div ton)$
and $\div$ is per_1
ton

But aren't we still missing something?

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Unlike *$100 per ton*, *cost per ton* seems to involve a division operation at a higher level, at the level of cost, creating a new measure function.

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Unlike *$100 per ton*, *cost per ton* seems to involve a division operation at a higher level, at the level of cost, creating a new measure function.

What does it mean to divide at the level of measure functions?

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Solution: Step 1

Start with quotient function *per*:

$per_0 \rightsquigarrow \lambda d \lambda q \, . \, \frac{q}{d}$ $\qquad \langle d, \langle d, d \rangle \rangle$

Lift type d to type $\langle e, d \rangle$ (the type of measure functions)

$per_{gg} \rightsquigarrow \lambda g_{\langle e,d \rangle} \lambda f_{\langle e,d \rangle} \lambda x \, . \, \frac{f(x)}{g(x)}$ $\qquad \langle ed, \langle ed, ed \rangle \rangle$

This is much like a Geach shift except that it's binary, lifting two arguments instead of one. It is also reminiscent of type polymorphism in coordination à la Partee and Rooth.

Solution: Step 2

Introduce a type-shifting operation that converts ‘ton’ to ‘ton of x’, or ‘the weight of x in tons’:

ton

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ton dimension: *weight*

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$\mapsto_{of}$

$$\lambda x\,.\,\frac{\text{weight}(x)}{\text{ton}}$$

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ton — dimension: *weight*

$\mapsto_{of}$

$\lambda x \, . \, \dfrac{\text{weight}(x)}{\text{ton}}$ — dimension: $\dfrac{\textit{weight}}{\textit{weight}} = 1$

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ton — dimension: *weight*

$$\mapsto_{of}$$

$$\lambda x\,.\,\frac{\text{weight}(x)}{\text{ton}}$$ — dimension: $\frac{weight}{weight} = 1$

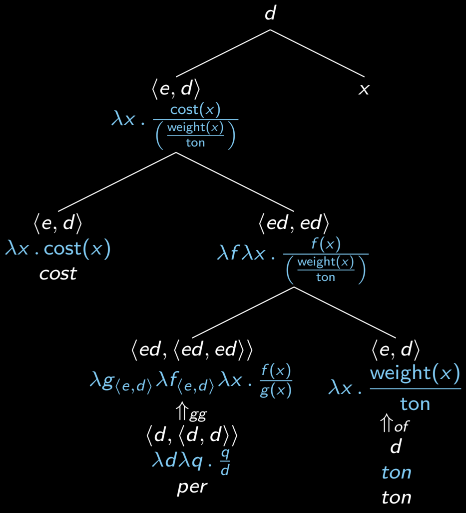
In general, for any unit u, we can shift it that way:

u

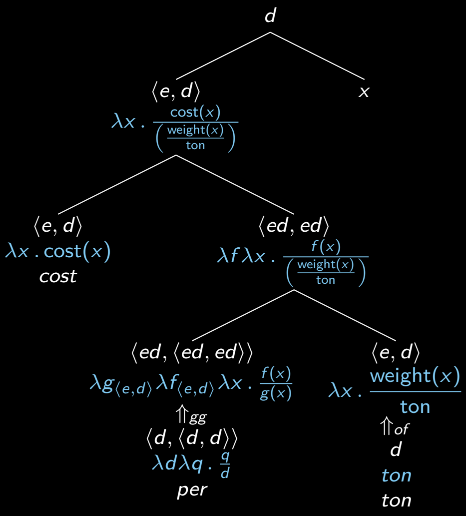
$$\mapsto_{of}$$

$$\lambda x\,.\,\frac{\mu_{\text{dim}(u)}(x)}{u}$$

The rest is automatic

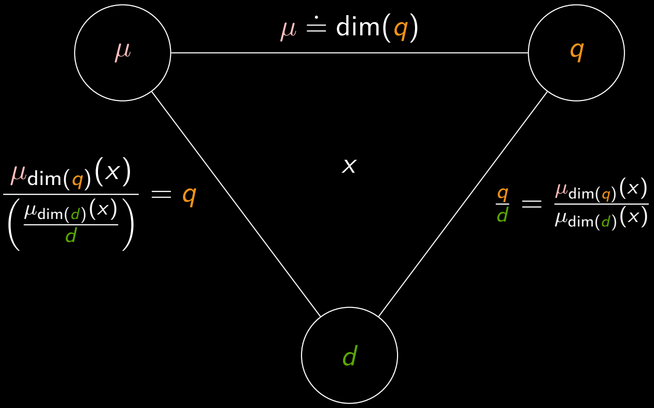


The rest is automatic



$$\frac{\mu_{\text{dim}(q)}(x)}{\left(\frac{\mu_{\text{dim}(d)}(x)}{d}\right)} = q$$

Now we have captured the intuition



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