Challenge Problems for a Theory of Degree Multiplication (with partial answer key)

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SALT 2021 · Brown University

Plan

Quotient dimensions

Proportional *few/many* (as opposed to cardinal):

(1) *Few egg-laying mammals suckle their young.*

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Degrees as proportions (Solt 2009, Bale & Schwarz 2019)?

Fractions in subscripts of measure-function μ in Bale and Schwarz's lexical entry for *much/many*:

(2) $\lambda d \lambda P \lambda Q \,.\, \mu(P \cap Q) \geq d$
where μ has a contextually set value, e.g. one of μ_{weight}, μ_{volume}, μ_{length}, $\mu_{\#}$, $\mu_{\frac{\text{weight}}{\text{vol-of-P}}}$, $\mu_{\frac{\text{weight}}{\text{vol-of-Q}}}$, $\mu_{\frac{\#}{\#\text{-of-P}}}$, $\mu_{\frac{\#}{\#\text{-of-Q}}}$, $\mu_{\frac{\#}{\text{length-of-rope}}}$, etc.

Percent

Conservativity-violating usages of *percent* (Ahn, 2012; Sauerland, 2014; Ahn & Sauerland, 2017; Sauerland & Pasternak, under review):

(3) *The company hired 30 percent women.*

Sauerland & Pasternak (under review) analyze *percent* as follows:

(4) $\lambda D_{dt} \lambda n_n \lambda D'_{dt} . D' \subseteq D \wedge \max(D') \geq \frac{n}{100} \times \max(D)$

See also Ahn & Sauerland (2015); Li (2018); Solt (2018); Spathas (2019); Pasternak (2019); Coppock (submitted).

Degree multiplication galore

Degree division

parts per million, *miles per hour*, *dollars per couple*, *hospitals per capita*
situps a day, *cents on the dollar*, *cents for every dollar*

Degree multiplication

A is twice as tall as B

cubic centimeters

3 apples at $2 per apple

Algebra of degrees: What we have

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But cross-dimensional multiplication and division will require more foundational changes.

Challenge problems

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Therefore, at 200 bpm it would be 1:48.

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Therefore, at 200 bpm it would be 1:48.

(10) *I bought this for $100 and sold it for 70 cents on the dollar.*
Therefore I sold it for $70.

Preview

kilometers per hour $\rightsquigarrow \frac{\text{km}}{\text{hour}}$

Quantity calculus: the study of quantities

quantity: property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference, e.g.:

- radius (of circle A), wavelength (of the sodium D radiation)
- kinetic energy, heat
- electric charge, electric resistance

(JCGM, 2012)

Quantity calculus

Three operations:

- product of quantities
- product of a number times a quantity
- addition of quantities of the same kind

(often presented as important starting point)

Quantity calculus

- History goes back to Fourier 1822 (de Boer, 1994)

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- Two approaches to the algebraic foundations:
  - **Unit-centric**: e.g. Carlson 1979, Kitano 2013
  - **Dimension-centric**: e.g. Krystek 2015, Raposo 2018, 2019

    "Under this viewpoint, the dimension is an intrinsic property of a quantity, in contrast to its numerical value, which depends on the unit chosen, or the unit itself, which can be changed arbitrarily."

Basic dimensions ($\mathscr{B}$)

Dimension	**Base unit**
L – length	meter (m)
M – mass	kilogram (kg)
T – time	second (s)
I – electric current	ampere (A)
Θ – thermodynamic temperature	kelvin (K)
N – amount of substance	mole (mol)
J – luminous intensity	candela (cd)

(JCGM, 2012)

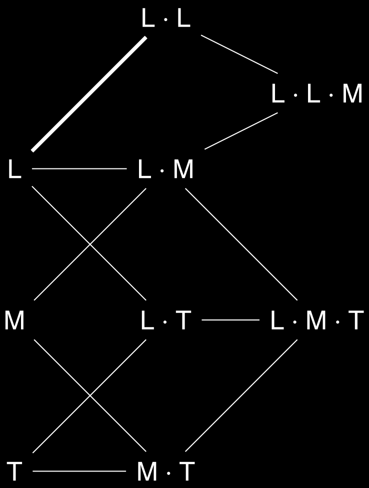
Derived dimensions

Example: If h is Planck's constant then

$$\mathsf{dim}(h) = \mathsf{M} \cdot \mathsf{L}^2 \cdot \mathsf{T}^{-1}$$

Fact:
The Planck constant multiplied by a photon's frequency is equal to a photon's energy.

Derived dimensions



The dimensions form a group

$\mathscr{D}$ is a group, so:

- if $\mathsf{A}, \mathsf{B} \in \mathscr{D}$, then $\mathsf{A} \cdot \mathsf{B} \in \mathscr{D}$
- $\mathscr{D}$ has an identity element $\mathbf{1}_{\mathscr{D}}$, such that for every $\mathsf{D} \in \mathscr{D}$:

$$\mathsf{D} \cdot \mathbf{1}_{\mathscr{D}} = \mathbf{1}_{\mathscr{D}} \cdot \mathsf{D} = \mathsf{D}$$

- There is an inverse D^{-1} for every $\mathsf{D} \in \mathscr{D}$:
  an element such that

$$\mathsf{D} \cdot \mathsf{D}^{-1} = \mathbf{1}_{\mathscr{D}}$$

Dimensionless quantities

So-called "dimensionless quantities" have dimension $\mathbf{1}_{\mathscr{D}}$:

- ratios of two quantities of the same kind
  - Ex. relative permeability, dollars earned per dollars saved
- numbers of entities
  - Ex. Number of molecules in a given sample

(JCGM, 2012)

Larger exponents

$$\mathsf{D}^0 = \mathbf{1}_{\mathscr{D}}$$
$$\mathsf{D}^1 = \mathsf{D}$$
$$\mathsf{D}^2 = \mathsf{D} \cdot \mathsf{D}$$
$$\mathsf{D}^3 = \mathsf{D} \cdot \mathsf{D}^{-2}$$
$$\vdots$$
$$\mathsf{D}^k = \mathsf{D} \cdot \mathsf{D}^{k-1}$$

$$\mathsf{D}^{-2} = (\mathsf{D}^{-1})^2$$
$$\mathsf{D}^{-3} = (\mathsf{D}^{-1})^3$$
$$\vdots$$
$$\mathsf{D}^{-k} = (\mathsf{D}^{-1})^k$$

Full set of dimensions

Each dimension $\mathsf{D} \in \mathscr{D}$ has a unique expression

$$\mathsf{D} = \mathsf{L}^{n_1} \cdot \mathsf{M}^{n_2} \cdot \mathsf{T}^{n_3} \cdot \mathsf{I}^{n_4} \cdot \Theta^{n_5} \cdot \mathsf{N}^{n_6} \cdot \mathsf{J}^{n_7}$$

where $n_1, ..., n_7$ are integers

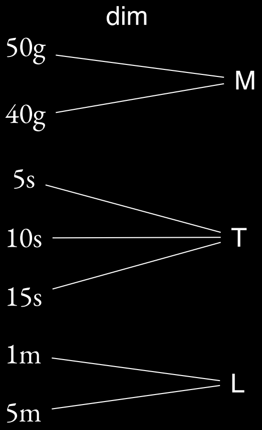
Dimension mapping

$$\mathscr{Q} \xrightarrow{\textsf{dim}} \mathscr{D}$$

for any quantity $\mathsf{Q} \in \mathscr{Q}$:

$$\textsf{dim}(\mathsf{Q}) \in \mathscr{D}$$

Dividing up the quantities by dimension



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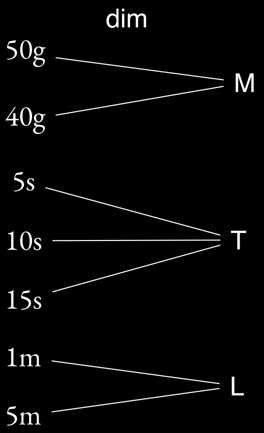

$\mathsf{dim}^{-1}(\mathsf{M})$
the set of quantities of mass

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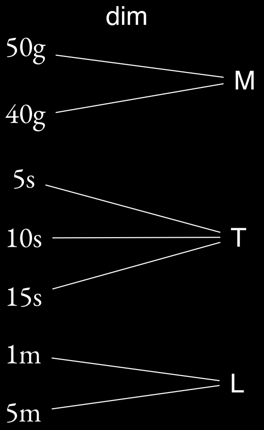

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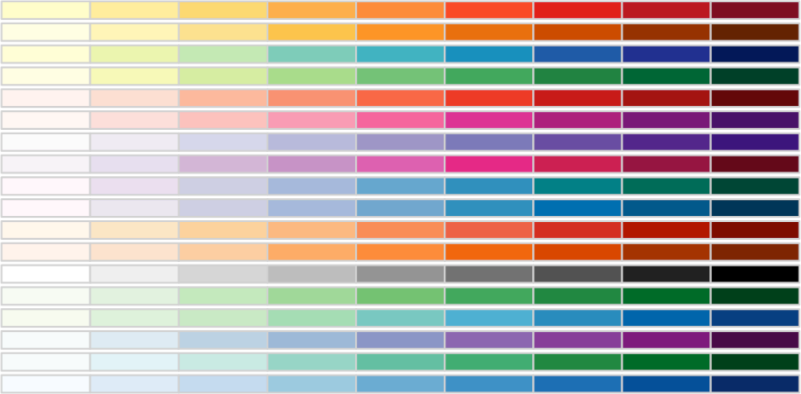
Notation:
$\mathsf{dim}^{-1}(\mathsf{D}) = \mathscr{Q}_{\mathsf{D}}$

Fiber bundle



Each fiber is a vector space over $\mathbb{R}$

For all dimensions D, $\mathscr{Q}_{\mathsf{D}}$ is a vector space over $\mathbb{R}$, so:

- There exists a **zero element** $\mathbf{0}_{\mathsf{D}} \in \mathscr{Q}_{\mathsf{D}}$ such that for any $q \in \mathscr{Q}_{\mathsf{D}}$:

$$q + \mathbf{0}_{\mathsf{D}} = q$$

- For any $q \in \mathscr{Q}_{\mathsf{D}}$, there exists an **additive inverse element** $-q \in \mathscr{Q}_{\mathsf{D}}$ such that:

$$q + (-q) = \mathbf{0}_{\mathsf{D}}$$

- There exists a **multiplicative identity element 1** from $\mathbb{R}$ such that for any $q \in \mathscr{Q}_{\mathsf{D}}$:

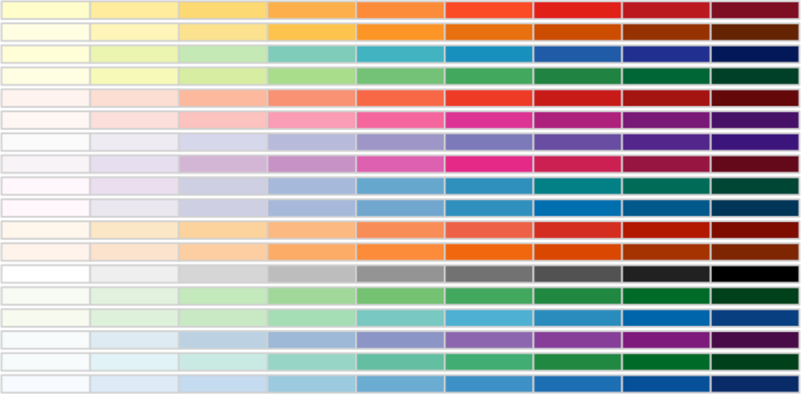
$$q * \mathbf{1} = \mathbf{1} * q = q$$

Each fiber is a vector space over $\mathbb{R}$

For all dimensions D, $\mathscr{Q}_{\mathsf{D}}$ is a vector space over $\mathbb{R}$,
so for any $q, q_1, q_2, q_3 \in \mathscr{Q}$ and **scalars** $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$:

- $q_1 + q_2 \in \mathscr{Q}_{\mathsf{D}}$ **(closure under addition)**
- $\alpha * q \in \mathscr{Q}_{\mathsf{D}}$ **(... scalar multiplication)**
- $q_1 + q_2 = q_2 + q_1$ **(commutativity of $+$)**
- $q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3$ **(associativity of $+$)**
- $\alpha_1 * (\alpha_2 * q) = (\alpha_1 \times \alpha_2) * q$ **(compatibility of $*$ and $\times$)**
- $\alpha * (q_1 + q_2) = \alpha * q_1 + \alpha * q_2$ **(distributivity 1)**
- $(\alpha_1 + \alpha_2) * q = \alpha_1 * q + \alpha_2 * q$ **(distributivity 2)**

Fiber bundle



Cross-dimensional multiplication

$\langle \mathcal{Q}, * \rangle$ is an **abelian monoid**, so:

- If $q_1, q_2 \in \mathcal{Q}$, then $q_1 * q_2 \in \mathcal{Q}$
- There is a **multiplicative identity** element **1** such that for all $q \in \mathcal{Q}$:

$$q * \mathbf{1} = \mathbf{1} * q = q$$

- If $q_1, q_2, q_3 \in \mathcal{Q}$ then
  $q_1 * (q_2 * q_3) = (q_1 * q_2) * q_3$ **(associativity)**
- $q_1 * q_2 = q_2 * q_1$ **(commutativity)**

Existence of inverses

Not every quantity has an inverse; you can't divide by any $\mathbf{0}_{\mathsf{D}}$ ($\mathsf{D} \in \mathscr{D}$).

But for every *non-zero* quantity $q \in \mathscr{Q}$
there is an inverse q^{-1}:

$$q * q^{-1} = \mathbf{1}$$

Or: The set of non-zero quantities forms a group under multiplication.

Dimension mapping

$$\mathscr{Q} \xrightarrow{\mathsf{dim}} \mathscr{D}$$

for any quantity $\mathsf{Q} \in \mathscr{Q}$: $\mathsf{dim}(\mathsf{Q}) \in \mathscr{D}$

Unit mapping

$$\mathscr{Q} \xleftarrow{\mathsf{unit}} \mathscr{D}$$

where $\mathsf{unit}(\mathsf{D})$ picks out a $q \in \mathscr{Q}_{\mathsf{D}}$
(a q such that $\mathsf{dim}(q) = \mathsf{D}$)

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Restrictions:

- You can't pick the zero element.
- $\textsf{unit}$ must be a group homomorphism:

$$\textsf{unit}(\textsf{A} \cdot \textsf{B}) = \textsf{unit}(\textsf{A}) * \textsf{unit}(\textsf{B})$$

Inverse units

Recall: Every non-zero $q \in \mathscr{Q}$ has an inverse q^{-1}.

So if $\mathsf{km} = \mathsf{unit}(\mathsf{L})$ and $\mathsf{hour} = 60 * (\mathsf{unit}(\mathsf{T}))$
then we can represent 'kilometers per hour' as:

$$\mathsf{km} * \mathsf{hour}^{-1}$$

Incorporating quantity calculus into a Montagovian framework

Representation language

$\mathcal{L}_{\mathscr{Q}}$: a lambda calculus with quantity multiplication.

The semantic value of an expression ϕ in $\mathcal{L}_{\mathscr{Q}}$ is given by $[\![\phi]\!]^M$, where:

$$M = \langle \mathcal{A}, \mathcal{V}, \langle \mathscr{D}_{\mathscr{B}}, \cdot \rangle, \langle \mathscr{Q}, *, + \rangle, \mathsf{unit}, \mathsf{dim}, I \rangle$$

where:

- $\mathcal{A}$ is a set of individuals, $\mathcal{V}$ a set of events
- $\langle \mathscr{D}_{\mathscr{B}}, \cdot \rangle$ is an abelian group with basis $\mathscr{B}$, a finite set of dimensions
- dim is a surjection map from $\mathscr{Q}$ onto $\mathscr{D}_{\mathscr{B}}$
- Each $\mathsf{dim}^{-1}(\mathsf{D}) = \mathscr{Q}_{\mathsf{D}}$ yields a one-dimensional vector space over $\mathbb{R}$
- $\langle \mathscr{Q}, * \rangle$ is an abelian monoid
- unit is a group homomorphism from $\mathscr{D}_{\mathscr{B}}$ to $\mathscr{Q}$
- I maps each constant of type τ to an element of D_τ

Importing the algebraic operations from the meta-language into the representation language:

- $[\![\alpha + \beta]\!]^M = [\![\alpha]\!]^M + [\![\beta]\!]^M$
- $[\![\alpha \cdot \beta]\!]^M = [\![\alpha]\!]^M \cdot [\![\beta]\!]^M$
- $[\![\alpha * \beta]\!]^M = [\![\alpha]\!]^M * [\![\beta]\!]^M$
- $[\![\alpha^{-n}]\!]^M = ([\![\alpha]\!]^M)^{-n}$

Abbreviation:

$$\alpha * \beta^{-1} \equiv \frac{\alpha}{\beta}$$

Denotations for some constants of type *d* in the representation language:

$[\![\mathsf{m}]\!]^M = \mathsf{m} = \mathsf{unit}(\mathsf{L})$
$[\![\mathsf{km}]\!]^M = 1000 * \mathsf{m}$

$[\![\mathsf{s}]\!]^M = \mathsf{s} = \mathsf{unit}(\mathsf{T})$
$[\![\mathsf{minute}]\!]^M = 60 * \mathsf{s}$
$[\![\mathsf{hour}]\!]^M = 60 * 60 * \mathsf{s}$

Lexical entries for English words:

(11) *meter(s)* $\rightsquigarrow$ m

(12) *kilometer(s)* $\rightsquigarrow$ km

(13) *second(s)* $\rightsquigarrow$ s

(14) *minute(s)* $\rightsquigarrow$ minute

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(16) *per* $\rightsquigarrow$ $\lambda d \lambda q \,.\, q * d^{-1}$

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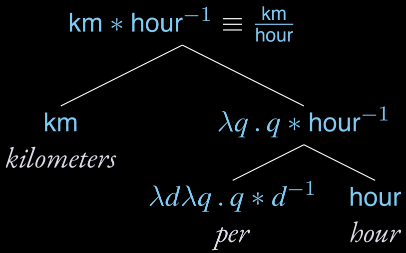
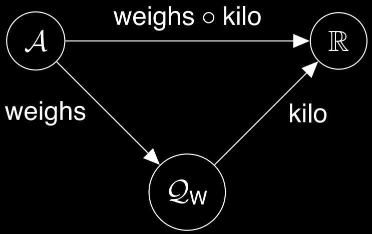


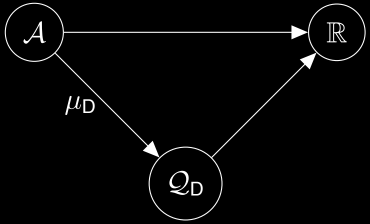
Linking individuals to quantities

The Lønning Triangle



(Lonning, 1987; Champollion, 2017)

The Lønning Triangle (our variant)



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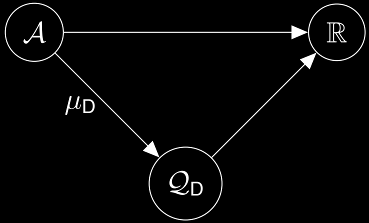

What is μ_{D}?

- The 'instantiates kind' relation? (Anderson & Morzycki, 2015; Scontras, 2017)
- The 'bears instance of trope type' relation? (Moltmann, 2009)

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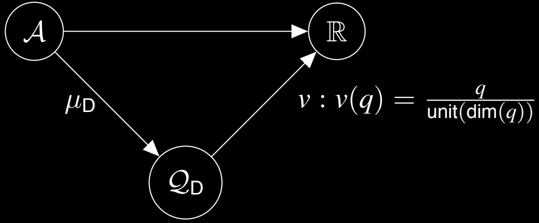

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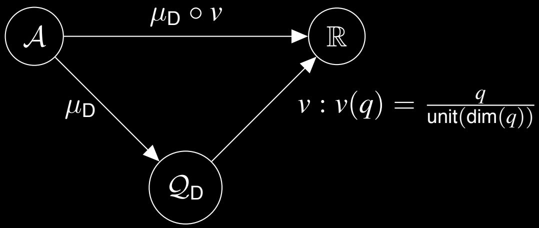

Recall: Bale & Schwarz notation

$\mu_{\text{weight}}, \mu_{\text{volume}}, \mu_{\text{length}}, \mu_{\#}, \mu_{\frac{\text{weight}}{\text{vol-of-P}}}, \mu_{\frac{\text{weight}}{\text{vol-of-Q}}}, \mu_{\frac{\#}{\text{\#-of-P}}}, \mu_{\frac{\#}{\text{\#-of-Q}}}, \mu_{\frac{\#}{\text{length-of-rope}}}$, etc.

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Now we can put some meat on the bones of formal representations like this; we have for example:

$$\mu_{\frac{\mathsf{L}}{\mathsf{T}}}$$

Challenge problems

Capturing inferences

(17) *Sainetra walked at 5 km per hour for 3 hours.*
Therefore, Sainetra walked 15 km.

Speed $*$ time $=$ distance

$$\mathsf{speed}(e) * \tau(e) = \sigma(e)$$

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Speed $*$ time $=$ distance

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$\mu_{\mathsf{T}} \equiv \tau$ ‘temporal extent’

$\mu_{\mathsf{L}} \equiv \sigma$ ‘spatial extent’

Speed $*$ time $=$ distance

$$\mathsf{speed}(e) * \tau(e) = \sigma(e)$$

$\mu_{\mathsf{T}} \equiv \tau$ ‘temporal extent’

$\mu_{\mathsf{L}} \equiv \sigma$ ‘spatial extent’

$\mu_{\frac{\mathsf{L}}{\mathsf{T}}} \equiv \mathsf{speed}$

Speed $*$ time $=$ distance

$$\mathsf{speed}(e) * \tau(e) = \sigma(e)$$

$\mu_{\mathsf{T}} \equiv \tau$ ‘temporal extent’

$\mu_{\mathsf{L}} \equiv \sigma$ ‘spatial extent’

$\mu_{\frac{\mathsf{L}}{\mathsf{T}}} \equiv \mathsf{speed}$

Axiom (or 'meaning postulate')

μ-**product principle:** The product of two measure functions is equal to the measure function along the product of the corresponding dimensions:

$$\mu_A(x) * \mu_B(x) = \mu_{A \cdot B}(x)$$

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Corollary:

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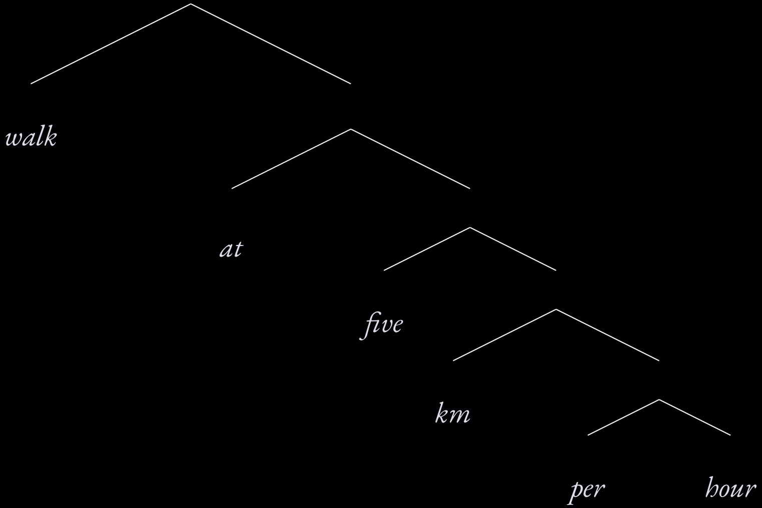
In particular:

$$\mu_{\frac{\mathsf{L}}{\mathsf{T}}}(x) * \mu_{\mathsf{T}}(x) = \mu_{\mathsf{L}}(x)$$


walk
at
five
km
per
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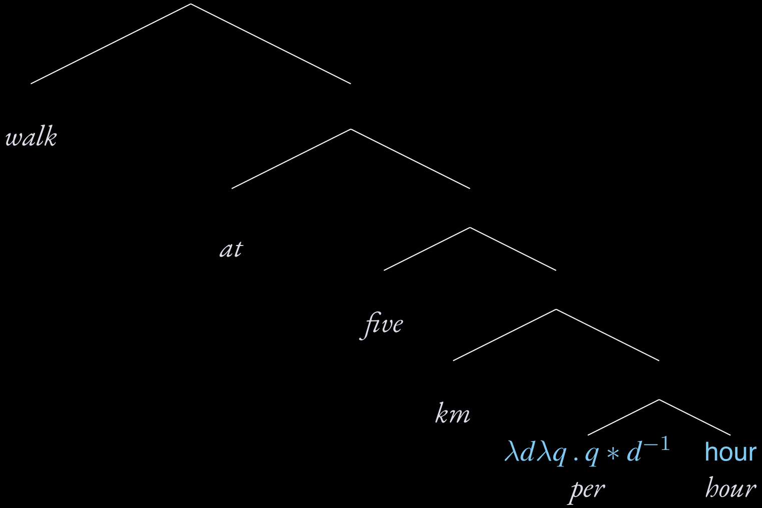
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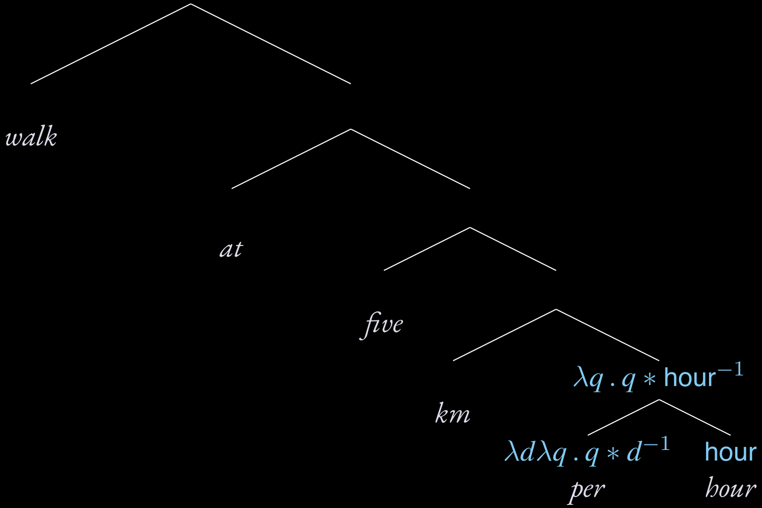
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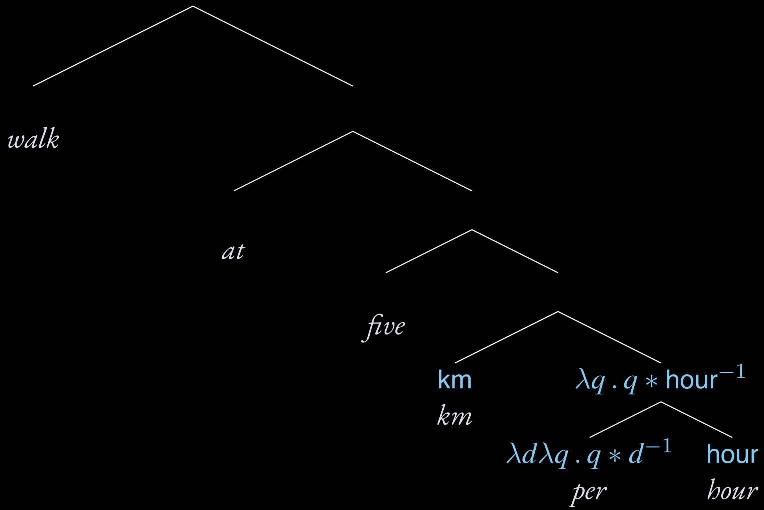
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λdλq . q ∗ d⁻¹
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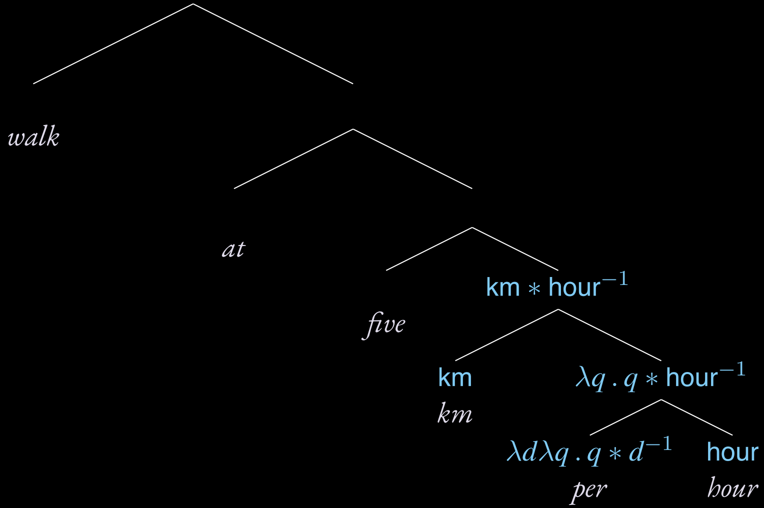
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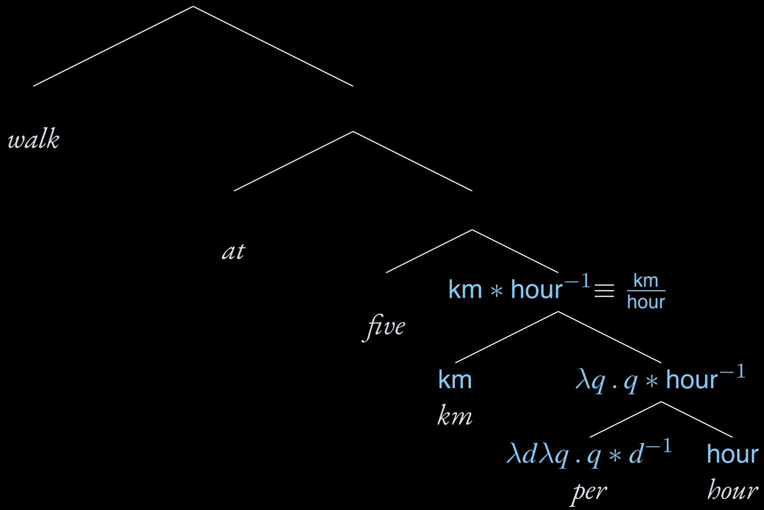
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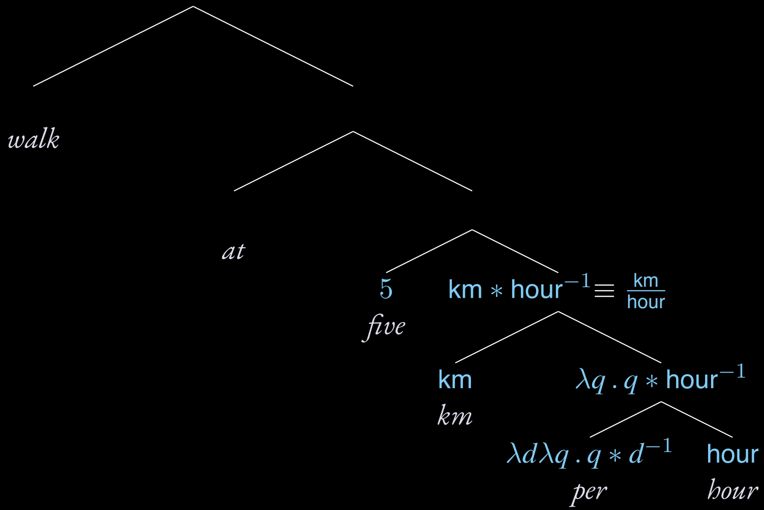
km ∗ hour^{-1} ≡ km/hour
km
km
λq . q ∗ hour^{-1}
λdλq . q ∗ d^{-1}
per
hour
hour


walk
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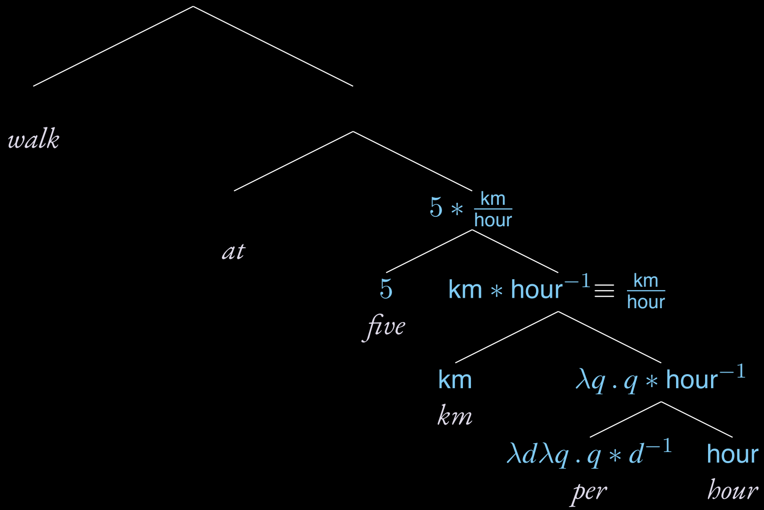
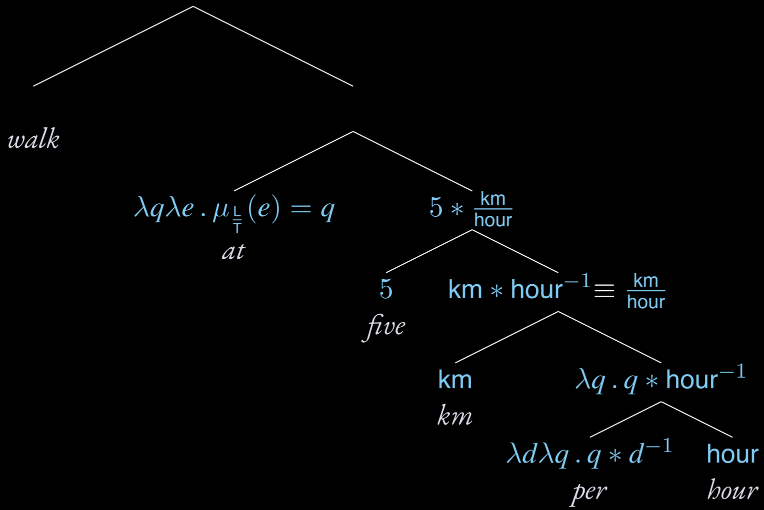
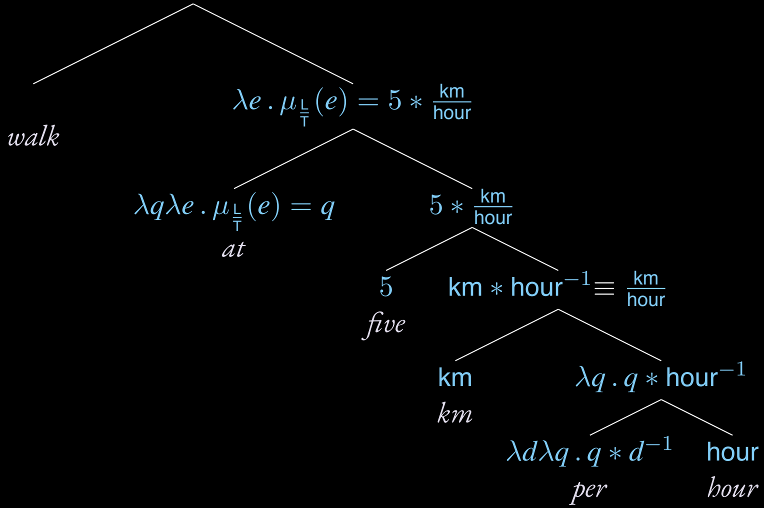


Cf. Ionin & Matushansky 2006


walk
$\lambda q \lambda e \,.\, \mu_{\frac{L}{T}}(e) = q$
at
$5 * \frac{\text{km}}{\text{hour}}$
5
five
$\text{km} * \text{hour}^{-1} \equiv \frac{\text{km}}{\text{hour}}$
km
km
$\lambda q \,.\, q * \text{hour}^{-1}$
$\lambda d \lambda q \,.\, q * d^{-1}$
per
hour
hour


walk
$\lambda e \,.\, \mu_{\frac{L}{T}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}}$
$\lambda q \lambda e \,.\, \mu_{\frac{L}{T}}(e) = q$
at
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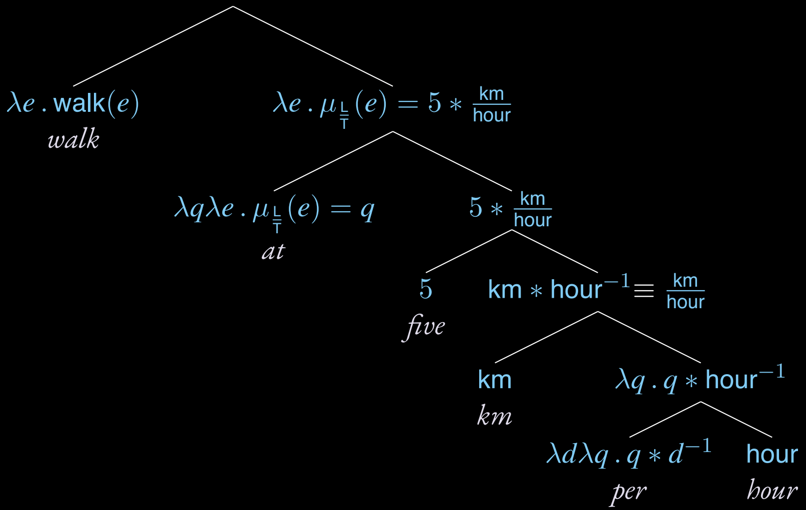
$\lambda e \,.\, \mathsf{walk}(e)$
walk
$\lambda e \,.\, \mu_{\frac{L}{T}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}}$
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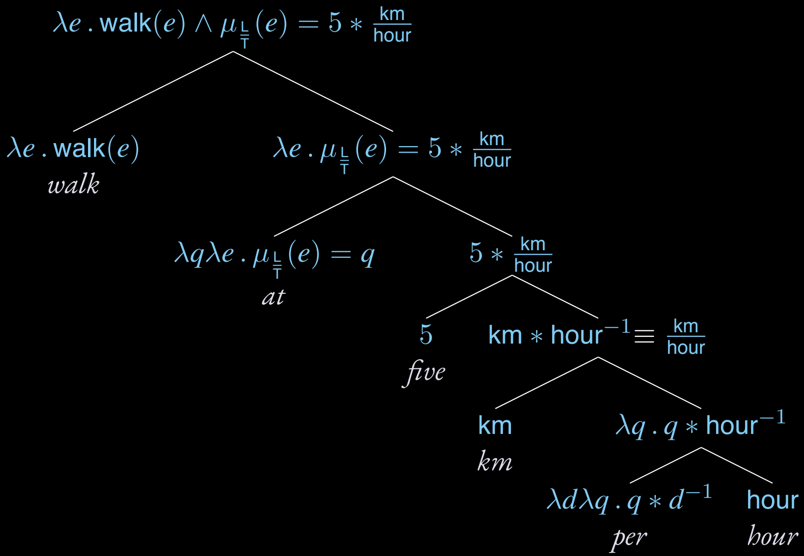
$\lambda e \,.\, \mathsf{walk}(e) \wedge \mu_{\frac{\mathsf{L}}{\mathsf{T}}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}}$
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walk
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Solution to challenge problem #1

walk at 5 km per hour $\rightsquigarrow \lambda e\,.\,\mathsf{Walk}(e) \wedge \mu_{\frac{\mathsf{L}}{\mathsf{T}}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}}$

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walk at 5 km per hour for three hours
$\rightsquigarrow \lambda e\,.\,\mathsf{Walk}(e) \wedge \mu_{\frac{\mathsf{L}}{\mathsf{T}}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}} \wedge \mu_{\mathsf{T}}(e) = 3 * \mathsf{hour}$

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By the μ-product principle, any event that satisfies the latter description also satisfies:

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By the μ-product principle, any event that satisfies the latter description also satisfies:

walk 15 km $\rightsquigarrow \lambda e\,.\,\mathsf{Walk}(e) \wedge \mu_{\mathsf{L}}(e) = 15 * \mathsf{km}$

Solution to challenge problem #1

walk at 5 km per hour $\rightsquigarrow \lambda e \,.\, \mathsf{Walk}(e) \wedge \mu_{\frac{\mathsf{L}}{\mathsf{T}}}(e) = 5 * \frac{\mathsf{km}}{\mathsf{hour}}$

walk at 5 km per hour for three hours
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By the μ-product principle, any event that satisfies the latter description also satisfies:

walk 15 km $\rightsquigarrow \lambda e \,.\, \mathsf{Walk}(e) \wedge \mu_{\mathsf{L}}(e) = 15 * \mathsf{km}$

Solved!

Challenge problem #2: Situps a day

(18) *Zahra did 30 situps a day for a week.*
Therefore, Zahra did 210 situps in one week.

Count dimensions

Option 1: Assume that for every property *P*, there is a different dimension #P.

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$\mu_{\#\text{bear}}(x) = 3 * \text{unit}(\#\text{bear})$ ‘*x* is three bears’

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$\mu_{\#\mathsf{bear}}(x) = 3 * \mathsf{unit}(\#\mathsf{bear})$ ‘*x* is three bears’

$\mu_{\#\mathsf{situp}}(e) = 30 * \mathsf{unit}(\#\mathsf{situp})$ ‘*e* is 30 situps’

Cf. Krifka’s (1995) ‘object unit’ function

What is a day?

Abbreviation:
$\mathsf{day} \equiv 24 * (60 * (60 * \mathsf{s}))$

This assumes 'day' = 'mean solar day'; otherwise we need two basic units of time!
Also, duration vs. object (Fillmore, 1997); cf. *every day* à la Champollion (2016a,b).

$$\textit{30 situps a day} \rightsquigarrow 30 * \frac{\text{unit}(\#\text{situps})}{\text{day}}$$

30 situps a day $\rightsquigarrow 30 * \frac{\text{unit}(\#\text{situps})}{\text{day}}$

do 30 situps a day $\rightsquigarrow \lambda e \,.\, \mu_{\frac{\#\text{situps}}{\text{T}}}(e) = 30 * \frac{\text{unit}(\#\text{situps})}{\text{day}}$

30 situps a day $\rightsquigarrow 30 * \frac{\mathsf{unit}(\#\mathsf{situps})}{\mathsf{day}}$

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do 30 situps a day for a week
$\rightsquigarrow \lambda e\,.\,\mu_{\frac{\#\mathsf{situps}}{\mathsf{T}}}(e) = 30 * \frac{\mathsf{unit}(\#\mathsf{situps})}{\mathsf{day}} \wedge \mu_{\mathsf{T}}(e) = 7 * \mathsf{day}$

30 situps a day $\rightsquigarrow 30 * \frac{\mathsf{unit}(\#\mathsf{situps})}{\mathsf{day}}$

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$$\mu_{\frac{\#\mathsf{situps}}{\mathsf{T}}}(x) * \mu_{\mathsf{T}}(x) = \mu_{\#\mathsf{situps}}(x)$$

30 situps a day $\rightsquigarrow 30 * \frac{\mathsf{unit}(\#\mathsf{situps})}{\mathsf{day}}$

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$$\mu_{\frac{\#\mathsf{situps}}{\mathsf{T}}}(x) * \mu_{\mathsf{T}}(x) = \mu_{\#\mathsf{situps}}(x)$$

do 210 situps in one week
$\rightsquigarrow \lambda e \,.\, \mu_{\#\mathsf{situps}}(e) = 210 * \mathsf{unit}(\#\mathsf{situps}) \wedge \mu_{\mathsf{T}}(e) = 7 * \mathsf{day}$

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Solved!*

Challenge problem #3: dollars per couple

(19) *Tickets cost $5 per couple.*
Therefore, tickets for 3 couples costs $15.

Tickets cost $5 per couple.

$$\forall x[\mathsf{tickets}(x) \rightarrow \mu_{\frac{\$\$}{\#\mathsf{couple}}}(x) = \frac{5 * \$}{\mathsf{unit}(\#\mathsf{couple})}]$$

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Tickets for 3 couples cost $15.

$$\forall x[[\textsf{tickets}(x) \wedge \mu_{\#\textsf{couple}}(x) = 3 * \textsf{unit}(\#\textsf{couple})] \rightarrow \mu_{\$\$}(x) = 15 * \$]$$

Tickets cost $5 per couple.

$$\forall x[\textsf{tickets}(x) \rightarrow \mu_{\frac{\$\$}{\#\textsf{couple}_{\textsf{Poss}}}}(x) = \frac{5*\$}{\textsf{unit}(\#\textsf{couple})}]$$

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$$\mu_{\#\mathsf{couple}_{\mathsf{Poss}}}(x) = 3 * \mathsf{unit}(\#\mathsf{couple}_{\mathsf{Poss}}) \equiv |y : \mathsf{couple}(y) \wedge \mathsf{Poss}(y, x)| = 3$$

Count dimensions: One or many?

Option #1:
For every P, a dimension $\#P$.

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Option #2:
There is only one # dimension.

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Option #2:
There is only one # dimension.

- Precedent in quantity calculus

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- Keeps $\mathscr{B}$ finite

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- How to distinguish:
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  - *tickets per person*

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For every P, a dimension $\#P$.

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Option #2:
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Takeaway messages

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- It's not something we have yet in degree semantics.
- We can get it by importing a system of quantity calculus.
- I have illustrated how to do this using the dimension-centric approach of Raposo (2018, 2019).
- At the minimum, we've gotten a lexical entry for *per*.
- But much more could be built on these foundations.

Thank you!

This work has benefitted from discussions with audience members at the NYU Philosophy of Language Workshop and the MIT Semantics Triangle, especially Lucas Champollion, Friederike Moltmann, Maša Močnik, Ying Gong, Hayley Ross, and Kai von Fintel. Thanks also to David Alvarez (BU undergraduate UROP research assistant, summer 2019) for a first pass on the analysis of *per*.

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